Representing Unawareness on State Spaces

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Abstract

We approach notions of unawareness in terms of the lack of knowledge within the framework of a standard state space model. Our questions are as follows. When and how does a state space model have a sensible form of unawareness? How does unawareness relate to ignorance and possibility? First, notions of unawareness reduce to the following two forms. A strong form of unawareness is stated as the ignorance of the possibility that an agent knows an event. A weak form of unawareness is stated as the ignorance of own knowledge. Second, we show that if an agent is unaware of an event, then she is ignorant of being unaware of it. Third, if an agent faces an infinite number of objects of knowledge, then it is possible that she knows that there is an event of which she is unaware, while she cannot know that she is unaware of any particular event. Fourth, getting more information can cause an agent to become unaware of some event.

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1 Introduction

Since the seminal work by Aumann [1, 2], a state space model of knowledge has been developed to model rational agents who reason interactively with each other. One of the subsequent research agendas in state space models has been to accommodate interactive knowledge among "boundedly rational" agents who lack some form of logical or introspective abilities. Especially, a notion of unawareness has been an active research area in economics since the pioneering work of Modica and Rustichini

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[24, 25] (hereafter, MR).[1] There, an agent is said to be unaware of a statement if she does not know it and she does not know that she does not know it.

Dekel, Lipman, and Rustichini [8] (hereafter, DLR), however, establish the following negative result. No state space model of knowledge can capture a sensible form of unawareness if an agent is logical and if a given notion of unawareness satisfies the following three axioms: Plausibility, KU Introspection, and AU Introspection. Plausibility states that if a player is unaware of an event, then she does not know it and she does not know that she does not know it. KU Introspection says that an agent does not know that she is unaware of any particular event. AU Introspection demands that if an individual is unaware of an event, then she is unaware of being unaware of that event.[2] Since then, the research focus on unawareness has shifted to scrutinizing concepts of unawareness and to developing enhanced models capable of representing non-trivial forms of unawareness satisfying desirable features including these three axioms.[3]

The purpose of this paper is to re-examine and to have a deeper understanding of how state space models can (and cannot) capture a sensible form of unawareness. To that end, we confine ourselves to imposing the following conditions on agents' knowledge and unawareness.

First, we assume that agents are logical and introspective as to their own knowledge. Namely, agents' knowledge satisfies at least the following three properties: (i) Truth Axiom (an agent can only know what is true), (ii) Positive Introspection (if an agent knows an event then she knows that she knows it), and (iii) Monotonicity (if an agent knows something then she knows its logical consequence). The first property is an essential property of knowledge, which distinguishes itself from belief. We assume the second and third properties because these are often assumed in state space models of knowledge involving boundedly rational agents (i.e., non-partitional (reflexive and transitive) possibility correspondence models of knowledge).[4]

Second, we define unawareness solely in terms of (the lack of) knowledge. We say that an agent is said to be k^n-unaware of an event if she does not know it, she does not know that she does not know it, and so forth n times, including the case of $n = \infty$. Thus, notions of unawareness are derived from given properties of knowledge and a level of the lack of knowledge.

Now, we ask the following two strands of questions. First, we ask conditions on knowledge under which the derived notions of unawareness have a non-trivial (or

[1] Other pioneering work include Fagin and Halpern [11] and Pires [27].

[2] MR [25] demonstrate another impossibility result when unawareness is symmetric (i.e., if a player is unaware of an event then she is also unaware of its negation).

[3] For example, Heifetz, Meier, and Schipper (hereafter, HMS) [16], [17], and Li [21] are earlier attempts along this line of research in economics. See also Schipper [31] for a recent overview.

[4] First, the previous studies on such models are: Bacharach [3], Binmore and Brandenburger [4], Brandenburger, Dekel, and Geanakoplos [5], Dekel and Gul [7], Geanakoplos [14], Morris [26], Rubinstein and Wolinsky [28], Samet [29, 30] and Shin [32]. Second, unlike possibility correspondence models, we do not necessarily impose Necessitation: an agent knows any form of tautology.

trivial) form in state space models. Especially, we aim to answer the question raised by DLR [8] and subsequently studied by Chen, Ely, and Luo (hereafter, CEL) [6]: which of the previous three axioms is to be retained to represent an interesting form of unawareness in a state space model?

The second strand of questions is to examine how the derived notions of unawareness are related to other notions derived from knowledge. Specifically, we investigate notions of possibility, knowing-whether, and ignorance. An agent considers an event E possible if she does not know its negation E^c (e.g., Hintikka [18]). An agent knows whether an event E is true when either she knows E or she knows its negation E^c (e.g., Hintikka [18] and Hart, Heifetz, and Samet (hereafter, HHS) [15]). An agent is said to be ignorant of an event E if she does not know E and she does not know its negation E^c (e.g., Lehrer and Samet [19]). That is, she is ignorant of E if shes does not know whether E is true.

Our results as follows. First, three levels of lack of knowledge imply any higher level of lack of knowledge. Thus, the derived notions of unawareness reduce to the two forms: either two levels of lack of knowledge or infinitely many levels of lack of knowledge. For each form of unawareness, we characterize it in terms of possibility and ignorance (Proposition 1). When unawareness is defined as two levels of lack of knowledge, we show that it is equivalent to the ignorance of own knowledge: an agent is k^2-unaware of an event E if and only if (hereafter, iff) she is ignorant of (not) knowing E. Next, we characterize infinitely many levels of lack of knowledge. We show that a player is k^∞-unaware of an event E iff she is ignorant of (not) knowing that she does not know E iff she is ignorant of the possibility that she knows E. We also show that a player is k^∞-unaware of an event E iff k^2-unaware of not knowing E.

Next, we give (in Proposition 2 and Corollary 1) a necessary and sufficient condition for a state space model to be non-trivial in terms of a qualitative feature of knowledge. A simple implication of this characterization is that any properly non-partitional state space model can capture a non-trivial form of unawareness if unawareness is defined as two levels of lack of knowledge.

What properties of unawareness do state space models satisfy? We show (in Proposition 3) that the notions of unawareness involve such properties as Plausibility, KU Introspection, the converse of AU Introspection, and the property which we call JU Introspection.[5] The converse of AU introspection refers to the property that if an agent is unaware of being unaware of an event E then she is unaware of E. JU Introspection is that if an agent is unaware of an event then she is ignorant of being unaware of it.

To restate, in any state space model where agents are logical and introspective and

[5]First, DLR [8, Footnote 10] note that a given notion of unawareness satisfies KU Introspection if it satisfies Plausibility and if a given notion of knowledge satisfies Truth Axiom and Monotonicity. Second, the "J" in JU Introspection refers to knowing-whether, and the symbol J is taken from the knowing-whether operator used in HHS [15].

where notions of unawareness are defined in terms of the lack of knowledge, the state space model satisfies Plausibility, KU Introspection, and JU Introspection (instead of AU Introspection). We also examine (in Proposition 4) properties of unawareness (e.g., AU Introspection and Symmetry) which lead to a degenerate form.

Finally, we study the following two properties of unawareness. First, recall that KU Introspection states that there is no state at which an agent knows that she is unaware of a particular event. Can an agent know her own unawareness? That is, can she know that there is an event of which she is unaware?[6] We show (in Proposition 5) that if there is an infinite number of objects of knowledge in a given state space model then an agent may know her self-unawareness, i.e., it can be the case that she knows there is an event of which she is unaware (while she does not know that she is unaware of any particular event). If objects of knowledge are finite in a given state space model, then an agent does not know that there is an event of which she is unaware.

Second, we provide examples in which unawareness is not monotonic in knowledgeability. Specifically, getting more information can cause an agent to become unaware of some event. We also study (in Propositions 6 and 7) the sense in which unawareness and knowledgeability are correlated.

The paper is organized as follows. Section 2 provides the state-space-based framework for studying interactive knowledge and unawareness. Section 3 studies properties of unawareness. In Section 3.1, we restate unawareness in terms of ignorance, knowing-whether, and possibility. In Section 3.2, we characterize a necessary and sufficient condition for non-trivial unawareness. Section 3.3 studies which of the existing axioms of unawareness are to be satisfied and violated in state space models.

Section 4 studies the following properties of unawareness. In Section 4.1, we ask knowledge of self-unawareness. In Section 4.2, we demonstrate non-monotonicity of unawareness in knowledgeability. In Section 4.3, we study possible forms of monotonicity of unawareness in knowledgeability. Proofs are relegated to Appendix A. Appendix B briefly discusses extensions of our framework.

2 Information Structures

In this section, we present our framework, which we call an information structure. Let Ω be a set of states (i.e., a *state space*), where each $\omega \in \Omega$ is referred to as a *state*. Let I denote a non-empty set of players. We call each $i \in I$ a *player*, an *individual*, or an *agent*.

Next, we define objects of interactive knowledge and unawareness which we will call events. For ease of exposition, let $\mathcal{D}$ be a complete algebra (of sets) on Ω. That is, $\mathcal{D}$ is a collection of events (i.e., a subset of $\mathcal{P}(\Omega)$, where $\mathcal{P}(\cdot)$ is the power set

[6] See also Schipper [31, Section 3.5] and the references therein.

operation) which is closed under complementation, arbitrary union, and arbitrary intersection.[7] We refer to each $E \in \mathcal{D}$ as an *event*. The collection $\mathcal{D}$ is called the *domain*.

Note that the specification of events (i.e., a domain) is an important consideration in establishing a "universal" structure and that we can indeed accommodate a specification of the domain $\mathcal{D}$ that guarantees existence of a universal structure (see Fukuda [13]). Namely, we can carry out most of the analyses when $\mathcal{D}$ is assumed to form a set algebra called a κ-complete algebra, where κ is an infinite cardinal.[8] With these definitions in mind, we now define an information structure.

Definition 1. *An information structure (of I) is a tuple $\mathcal{S} := \langle(\Omega, \mathcal{D}), (K_i, U_i)_{i\in I}\rangle$ with the following properties.*

1. *Ω is a state space, and $\mathcal{D}$ is a complete algebra of events.*
2. *$K_i : \mathcal{D} \to \mathcal{D}$ is player i's knowledge operator satisfying (at least) the following.*
   (a) *Truth Axiom: $K_i(E) \subseteq E$ (for all $E \in \mathcal{D}$).*
   (b) *Positive Introspection: $K_i(E) \subseteq K_iK_i(E)$.*
   (c) *Monotonicity: if $E \subseteq F$ then $K_i(E) \subseteq K_i(F)$.*
3. *$U_i : \mathcal{D} \to \mathcal{D}$ is player i's unawareness operator.*

Fix any event $E \in \mathcal{D}$. The set $K_i(E)$ is the event that (the set of states at which) i knows E. Likewise, $U_i(E)$ is the event that player i is unaware of E.

We maintain the common domain assumption, although we can possibly allow distinct domains $\mathcal{D}_i(\subseteq \mathcal{P}(\Omega))$ of K_i and U_i across players. We briefly discuss such player-specific domains in Appendix B. One possible justification of the common domain assumption is that we restrict attention to those events in $\mathcal{D} = \bigcap_{i\in I} \mathcal{D}_i$ under heterogeneous domains.

Truth Axiom distinguishes knowledge from "belief" in the sense that an agent can only know what is true while she can believe what is false. Positive Introspection allows an agent to know what she knows. Monotonicity renders an agent a logical inference ability.

Further properties of knowledge operators are given as follows. First, K_i satisfies Necessitation if $K_i(\Omega) = \Omega$. Second, K_i satisfies Non-empty Conjunction if $\bigcap_{E\in\mathcal{E}} K_i(E) \subseteq K_i(\bigcap \mathcal{E})$ for any non-empty $\mathcal{E} \subseteq \mathcal{D}$. In a similar vein, we define Non-empty Finite (Countable) Conjunction as follows: $\bigcap_{E\in\mathcal{E}} K_i(E) \subseteq K_i(\bigcap \mathcal{E})$ for any

[7] If $\mathcal{E}$ is a subset of $\mathcal{D}$, then $\bigcup \mathcal{E}(:= \bigcup_{E\in\mathcal{E}} E) \in \mathcal{D}$ and $\bigcap \mathcal{E}(:= \bigcap_{E\in\mathcal{E}} E) \in \mathcal{D}$. Also, if $E \in \mathcal{D}$ then $E^c \in \mathcal{D}$, where E^c is the complement of E (we also use $\neg E$ to denote the complement of E). Note that we follow the conventions that $\emptyset = \bigcup \emptyset$ and that, with an underlying set Ω fixed, $\Omega = \bigcap \emptyset$.

[8] First, $\mathcal{D}$ is a κ-complete algebra if the following hold: if $E \in \mathcal{D}$ then $E^c \in \mathcal{D}$; and if $\mathcal{E}$ satisfies $\mathcal{E} \subseteq \mathcal{D}$ and $|\mathcal{E}| < \kappa$ then $\bigcap \mathcal{E} \in \mathcal{D}$ and $\bigcup \mathcal{E} \in \mathcal{D}$. Second, if κ is the least infinite cardinal, we would have to take care of infinite operations in the analyses to follow.

non-empty finite (countable) $\mathcal{E} \subseteq \mathcal{D}$. Note that Non-empty Conjunction and Necessitation can be jointly regarded as (Arbitrary) Conjunction: $\bigcap_{E \in \mathcal{E}} K_i(E) \subseteq K_i(\bigcap \mathcal{E})$ for any $\mathcal{E} \subseteq \mathcal{D}$. Third, we define Negative Introspection of K_i to be $(\neg K_i)(\cdot) \subseteq K_i(\neg K_i)(\cdot)$.

Next, we state some of joint postulates on players' knowledge and unawareness operators. First, we say that (K_i, U_i) is *plausible* if $U_i(\cdot) \subseteq (\neg K_i)(\cdot) \cap (\neg K_i)^2(\cdot)$. Plausibility says that if an agent is unaware of an event E then she does not know E and she does not know that she does not know E. If every (K_i, U_i) is plausible, we say that $\mathcal{S}$ is plausible.

Second, we say that (K_i, U_i) satisfies *KU Introspection* if $K_i U_i(\cdot) = \emptyset$. KU Introspection means that, for any event E, there is no state at which an agent knows that she is unaware of E. Third, we say that (K_i, U_i) satisfies *AU Introspection* if $U_i(\cdot) \subseteq U_i U_i(\cdot)$. AU Introspection says that if an agent is unaware of E then she is unaware of being unaware of E. These three properties are proposed in DLR [8]. Other properties are examined in Section 3.3.

To conclude the exposition of the framework, we relate this framework to possibility correspondence models. A knowledge operator K_i satisfying Monotonicity and Arbitrary Conjunction can be induced from a possibility correspondence. If K_i also satisfies Truth Axiom and Positive Introspection, then it is characterized by a reflexive and transitive possibility correspondence. See Footnote 4 for the literature. If K_i additionally satisfies Negative Introspection, then it is induced by a partition (Aumann [1, 2]). In any plausible information structure $\mathcal{S}$ such that K_i is induced from a partition, however, we have $U_i(\cdot) = \emptyset$.

2.1 Associated Concepts

We define other associated concepts which are derived from knowledge (and unawareness). In this subsection, fix an information structure $\mathcal{S}$ of I.

Derived Operators. We start with defining the following four operators defined on $\mathcal{D}$. For any event $E \in \mathcal{D}$, we let: (i) $L_i(E) := (\neg K_i)(E^c)$, (ii) $\partial_i(E) := (\neg K_i)(E) \cap (\neg K_i)(E^c)$, (iii) $J_i(E) := (\neg \partial_i)(E) (= K_i(E) \cup K_i(E^c))$, and (iv) $A_i(E) := (\neg U_i)(E)$. We call L_i, ∂_i, J_i, and A_i to be *possibility* (e.g., Hintikka [18]), *ignorant* (e.g., Lehrer and Samet [19]), *knowing-whether* (e.g., Hintikka [18] and HHS [15]), and *awareness* (MR [24, 25]) operators, respectively.

First, $L_i(E)$ is regarded as the event that i considers E possible in the sense that i does not know its negation E^c. Second, $\partial_i(E)$ is interpreted as the event that i is ignorant of E in the sense that i does not know E nor E^c.[9] Third, $J_i(E)$ is considered to be the event that i knows whether E obtains (or not) in the sense that either i knows E or she knows its negation E^c. Fourth, $A_i(E)$ is regarded as the event that i

[9] We use the symbol "∂" of the boundary operator in a topological space in the sense that K_i satisfies a part of the properties satisfied by the interior operator in a topological space.

is aware of E in the sense that i is not unaware of E.

Defining Unawareness from Knowledge. We define unawareness operators derived from a given knowledge operator as follows. For $n \in \mathbb{N}_2^\infty := \{n \in \mathbb{N} \mid n \geq 2\} \cup \{\infty\}$, we define $U_i^{(n)}(\cdot) := \bigcap_{r=1}^n (\neg K_i)^r(\cdot)$.[10] We say that a player i is (k^n-)*unaware* of an event E at a state ω if $\omega \in U_i^{(n)}(E)$. The k^n-awareness operator $A_i^{(n)}$ is defined by $A_i^{(n)}(\cdot) := (\neg U_i^{(n)})(\cdot)$. MR [24] define their unawareness operator by $U_i^{(2)}$ while $U_i^{(\infty)}$ is considered in DLR [8].

Self-evident Collection. We define an event $E \in \mathcal{D}$ to be *self-evident* to a player i if $E \subseteq K_i(E)$. We denote by $\mathcal{J}_i := \{E \in \mathcal{D} \mid E \subseteq K_i(E)\}$ the collection of events which are self-evident to i. We call $\mathcal{J}_i$ to be i's *self-evident collection*. We have $\mathcal{J}_i \neq \emptyset$ because $\emptyset \in \mathcal{J}_i$. Also, it can be seen that Monotonicity of K_i implies that $\mathcal{J}_i$ is closed under arbitrary union.[11] Denoting $\mathcal{A}_i := \{E \in \mathcal{D} \mid L_i(E) \subseteq E\}$, we have $\mathcal{A}_i = \{E \in \mathcal{D} \mid E^c \in \mathcal{J}_i\}$.

Conversely, given the self-evident collection $\mathcal{J}_i$ which is induced from K_i, we can define the associated knowledge operator $K_{\mathcal{J}_i}(E) := \{\omega \in \Omega \mid \omega \in F \subseteq E \text{ for some } F \in \mathcal{J}_i\} = \bigcup\{F \in \mathcal{J}_i \mid F \subseteq E\}$. It turns out that $K_i = K_{\mathcal{J}_i}$. Moreover, the following can be verified (see Fukuda [13] for proofs). First, K_i satisfies Non-empty (Finite/Countable) Conjunction iff $\mathcal{J}_i$ is closed under non-empty (finite/countable) intersection. Second, K_i satisfies Necessitation iff $\Omega \in \mathcal{J}_i$. Third, K_i satisfies Negative Introspection iff $\mathcal{J}_i$ is closed under complementation.

Knowledgeability. We denote by $\mathrm{IK}_i(\omega) := \{E \in \mathcal{D} \mid \omega \in K_i(E)\}$ the collection of events that a player i knows at a state ω. We say that an individual i is *at least as knowledgeable as* an individual j *at* a state ω if $\mathrm{IK}_j(\omega) \subseteq \mathrm{IK}_i(\omega)$. We also say that individuals i and j are *equally knowledgeable at* a state ω if $\mathrm{IK}_j(\omega) = \mathrm{IK}_i(\omega)$.

Likewise, we say that an individual i is *at least as knowledgeable as* an individual j if i is at least as knowledgeable as j at any state. We also say that individuals i and j are *equally knowledgeable* if i and j are equally knowledgeable at any state. It can be easily seen that this knowledgeable relation is formulated in terms of knowledge operators and self-evident collections: i is at least as knowledgeable as j iff $K_j(\cdot) \subseteq K_i(\cdot)$ iff $\mathcal{J}_j \subseteq \mathcal{J}_i$.

Common Knowledge. We define the notion of common knowledge (e.g., Aumann [1], Friedell [12], Lewis [20], and McCarthy, Sato, Hayashi, and Igarashi [22]) among I as the knowledge that would be possessed by the most knowledgeable individual who is at least as less knowledgeable as every individual player within I.[12] We say

[10]Note that $\mathbb{N}$ is the set of positive integers and that $U_i^{(\infty)}(\cdot) = \bigcap_{r \in \mathbb{N}} (\neg K_i)^r(\cdot)$.

[11]In mathematical psychology, Doignon and Falmagne [9, 10] formalize a notion of knowledge by a set algebra, which is related to a self-evident collection. See Fukuda [13] for the discussion.

[12]Aumann [1] defines the notion of common knowledge from the finest partition which is coarser

that an event $E \in \mathcal{D}$ is *commonly known/common knowledge* among I at a state $\omega \in \Omega$ if E is inferred from some event F in the sense of $F \subseteq E$, where F is true at ω and where F is self-evident to every player $i \in I$ (i.e., *publicly evident* (Milgrom [23])). Formally, we define the *common knowledge operator* $C_I : \mathcal{D} \to \mathcal{D}$ by $C_I(E) := \{\omega \in \Omega \mid \omega \in F \subseteq E \text{ for some } F \in \bigcap_{i \in I} \mathcal{J}_i\} = \bigcup\{F \in \bigcap_{i \in I} \mathcal{J}_i \mid F \subseteq E\}$.[13] It can be seen that C_I inherits the properties that every individual knowledge operator satisfies, because taking the intersection of $(\mathcal{J}_i)_{i \in I}$ preserves set-algebraic properties that every $\mathcal{J}_i$ commonly possess.[14]

If E is commonly known among I at ω, then everyone in the group I knows E at ω, everyone in I knows that everyone in I knows E at ω, *ad infinitum*. Let $K_I(\cdot) := \bigcap_{i \in I} K_i(\cdot)$. That is, $K_I(E)$ is the event that everyone in I knows E. Indeed, it can be seen that $C(\cdot) = \bigcap_{n \in \mathbb{N}} K_I^n(\cdot)$ in any information structure satisfying Countable Conjunction, as $\bigcap_{n \in \mathbb{N}} K_I^n(E)$ is the maximal publicly evident event contained in E (the fact that it is publicly evident follows because, $\bigcap_{n \in \mathbb{N}} K_I^n(E) \subseteq \bigcap_{n \in \mathbb{N}} K_I^{n+1}(E) \subseteq \bigcap_{n \in \mathbb{N}} K_i K_I^n(E) = K_i(\bigcap_{n \in \mathbb{N}} K_I^n(E))$ for all $i \in I$ under Countable Conjunction).

Common (Un)awareness. We define the *common awareness* operator (HMS [17]) $CA_I : \mathcal{D} \to \mathcal{D}$ by $CA_I(\cdot) := \bigcap_{r \in \mathbb{N}} A_I^r(\cdot)$, where A_I is a *mutual awareness* operator defined by $A_I(\cdot) := \bigcap_{i \in I} A_i(\cdot)$. By definition, $K_I(\cdot) \subseteq A_I(\cdot)$. Also, Monotonicity of K_I implies that $K_I^r(\cdot) \subseteq A_I^r(\cdot)$ for each $r \in \mathbb{N}$, and hence we have $C_I(\cdot) \subseteq CA_I(\cdot)$.

Likewise, we define the *common unawareness* operator $CU_I : \mathcal{D} \to \mathcal{D}$ by $CU_I(\cdot) := \bigcap_{r \in \mathbb{N}} U_I^r(\cdot)$, where U_I is the *mutual unawareness* operator defined by $U_I(\cdot) := \bigcap_{i \in I} U_i(\cdot)$. It is clear that $U_I(\cdot) \subseteq (\neg C_I)(\cdot)$.

2.2 An Example

We provide an example to illustrate our framework.

Example 1. We let $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathcal{D} = \mathcal{P}(\Omega)$, and $I = \{i_1, i_2, i_3, i_4\}$. Suppose that each K_i is defined as in Table 1. Each $U_i^{(n)}$ (with $n \in \mathbb{N}_2^{\infty}$) is depicted in Table 1. Note that player i_1's knowledge coincides with DLR's [8, p. 161] Example 1.

We make the following remarks. First, we have $\mathcal{J}_{i_1} = \{\emptyset, \{\omega_1\}, \{\omega_2\}, \{\omega_1, \omega_2\}, \Omega\}$, $\mathcal{J}_{i_2} = \{\emptyset, \{\omega_1\}, \{\omega_3\}, \{\omega_1, \omega_3\}\}$, $\mathcal{J}_{i_3} = \{\emptyset, \{\omega_1\}, \Omega\}$, and $\mathcal{J}_{i_4} = \{\emptyset, \{\omega_1\}\}$. Thus, player i_1 is at least as knowledgeable as $j \in \{i_2, i_3, i_4\}$.

than every player's partition.

[13] We can similarly define the common knowledge operator C_G for any $G \in \mathcal{P}(I)$ with the convention that $C_\emptyset(E) = E$ for all $E \in \mathcal{D}$.

[14] Hence, we can apply the concept of k^n-unawareness to the common knowledge operator. We can say that an event E is n-negatively common knowledge among I at a state ω if $\omega \in U_{C_I}^{(n)}(E) := \bigcap_{r=1}^{n} (\neg C_I)^r(E)$ for each $n \in \mathbb{N}_2^{\infty}$. Hence, an event E is twice negatively common knowledge iff E is not common knowledge and the event that E is not common knowledge is not common knowledge.

E	K_{i_1}	$(\neg K_{i_1})$	$(\neg K_{i_1})^2$	$(\neg K_{i_1})^3$	$(\neg K_{i_1})^4$	∂_{i_1}	$U_{i_1}^{(2)}$	$U_{i_1}^{(n)}$
$\emptyset$	$\emptyset$	Ω	$\emptyset$	Ω	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_3\}$	$\{\omega_3\}$	$\{\omega_3\}$
$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_3\}$	$\{\omega_3\}$	$\{\omega_3\}$
$\{\omega_3\}$	$\emptyset$	Ω	$\emptyset$	Ω	$\emptyset$	$\{\omega_3\}$	$\emptyset$	$\emptyset$
$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_3\}$	Ω	$\emptyset$	Ω	$\{\omega_3\}$	$\{\omega_3\}$	$\emptyset$
$\{\omega_1, \omega_3\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_3\}$	$\{\omega_3\}$	$\{\omega_3\}$
$\{\omega_2, \omega_3\}$	$\{\omega_2\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_3\}$	$\{\omega_3\}$	$\{\omega_3\}$
Ω	Ω	$\emptyset$	Ω	$\emptyset$	Ω	$\emptyset$	$\emptyset$	$\emptyset$

E	K_{i_2}	$(\neg K_{i_2})$	$(\neg K_{i_2})^2$	$(\neg K_{i_2})^3$	$(\neg K_{i_2})^4$	∂_{i_2}	$U_{i_2}^{(2)}$	$U_{i_2}^{(n)}$
$\emptyset$	$\emptyset$	Ω	$\{\omega_2\}$	Ω	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$
$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$
$\{\omega_2\}$	$\emptyset$	Ω	$\{\omega_2\}$	Ω	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$
$\{\omega_3\}$	$\{\omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$
$\{\omega_1, \omega_2\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$
$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2\}$	Ω	$\{\omega_2\}$	Ω	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$
$\{\omega_2, \omega_3\}$	$\{\omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$
Ω	$\{\omega_1, \omega_3\}$	$\{\omega_2\}$	Ω	$\{\omega_2\}$	Ω	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$

E	K_{i_3}	$(\neg K_{i_3})$	$(\neg K_{i_3})^2$	$(\neg K_{i_3})^3$	$(\neg K_{i_3})^4$	∂_{i_3}	$U_{i_3}^{(2)}$	$U_{i_3}^{(n)}$
$\emptyset$	$\emptyset$	Ω	$\emptyset$	Ω	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	Ω	$\emptyset$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\emptyset$
$\{\omega_2\}$	$\emptyset$	Ω	$\emptyset$	Ω	$\emptyset$	$\{\omega_2, \omega_3\}$	$\emptyset$	$\emptyset$
$\{\omega_3\}$	$\emptyset$	Ω	$\emptyset$	Ω	$\emptyset$	$\{\omega_2, \omega_3\}$	$\emptyset$	$\emptyset$
$\{\omega_1, \omega_2\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	Ω	$\emptyset$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\emptyset$
$\{\omega_1, \omega_3\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	Ω	$\emptyset$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\emptyset$
$\{\omega_2, \omega_3\}$	$\emptyset$	Ω	$\emptyset$	Ω	$\emptyset$	$\{\omega_2, \omega_3\}$	$\emptyset$	$\emptyset$
Ω	Ω	$\emptyset$	Ω	$\emptyset$	Ω	$\emptyset$	$\emptyset$	$\emptyset$

E	$C_I = K_{i_4}$	$(\neg K_{i_4})$	$(\neg K_{i_4})^2$	$(\neg K_{i_4})^3$	$(\neg K_{i_4})^4$	∂_{i_4}	$U_{i_4}^{(2)}$	$U_{i_4}^{(n)}$
$\emptyset$	$\emptyset$	Ω	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$
$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$
$\{\omega_2\}$	$\emptyset$	Ω	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$
$\{\omega_3\}$	$\emptyset$	Ω	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$
$\{\omega_1, \omega_2\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$
$\{\omega_1, \omega_3\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$
$\{\omega_2, \omega_3\}$	$\emptyset$	Ω	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$
Ω	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$

Table 1: Illustrations of Players' Knowledge and Unawareness in Example 1 ($n \in \{n \in \mathbb{N} \mid n \geq 3\} \cup \{\infty\}$)

Second, each K_i satisfies Non-empty Conjunction. Knowledge operators of i_1 and i_3 also satisfy Necessitation.

Third, the collection of publicly evident events is $\bigcap_{i \in I} \mathcal{J}_i = \{\emptyset, \{\omega_1\}\} = \mathcal{J}_{i_4}$, so that we can also identify player i_4 as a dummy player whose knowledge corresponds with common knowledge (i.e., $C_I = K_{i_4}$).

Fourth, each pair $(K_i, U_i^{(n)})$ satisfies Plausibility and KU Introspection. While $(K_i, U_i^{(n)})$ satisfies AU Introspection for each $i \in \{i_2, i_4\}$, the other pairs $(K_j, U_j^{(n)})$ ($j \in \{i_1, i_3\}$) do not satisfy AU introspection because $U_j^{(n)} U_j^{(n)}(\cdot) = \emptyset$. Such j's information structure $\langle (\Omega, \mathcal{D}), (K_j, U_j^{(n)}) \rangle$ ($n \in \mathbb{N}_2^{\infty}$) is considered to be a reflexive and transitive possibility correspondence model. Indeed, j's knowledge can be induced from the possibility correspondence $b_j : \Omega \to \mathcal{D}$, where $b_{i_1}(\omega_1) = \{\omega_1\}$, $b_{i_1}(\omega_2) = \{\omega_2\}$, and $b_{i_1}(\omega_3) = \Omega$; and $b_{i_3}(\omega_1) = \{\omega_1\}$ and $b_{i_3}(\omega_2) = b_{i_3}(\omega_3) = \Omega$. The fact that $\langle (\Omega, \mathcal{D}), (K_j, U_j^{(n)}) \rangle$ does not satisfy AU Introspection is consistent with the finding by DLR [8] that there is no possibility correspondence model which satisfies all of Plausibility, KU Introspection, and AU Introspection.

Fifth, $U_I^{(n)}(\cdot) = \emptyset$ so that $CU_I(\cdot) = \emptyset$. Also, $A_I^{(n)}(\cdot) = \{\omega_1\}$ so that $CA_I(\cdot) = \{\omega_1\}$.

Sixth, player i_2's (resp. i_4's) knowledge operator can be identified as being defined on $\{\omega_1, \omega_3\}$ (resp. $\{\omega_1\}$). In other words, any state $\omega \in \{\omega_2\}$ (resp. $\omega \in \{\omega_2, \omega_3\}$) is deemed "impossible" by player i_2 (resp. i_4). On the other hand, any event $E \in \mathcal{P}(\{\omega_1, \omega_3\})$ (resp. $E \in \mathcal{P}(\{\omega_1\})$) is self-evident to player i_2 (resp. i_4). In this example, their unawareness is always determined by "impossible" states.

3 Unawareness on State Spaces

Having defined the basic framework, we now proceed with the main analyses.

3.1 Equivalent Representations

Our first aim is to relate the concepts of ignorance, knowing-whether, and possibility to that of unawareness when they are derived from knowledge. We also study the relations among the notions of k^n-unawareness. To that end, throughout the subsection, we fix an information structure $\mathcal{S} = \langle (\Omega, \mathcal{D}), (K, U) \rangle$ of a single player.

The first benchmark result is that, under Truth Axiom, Positive Introspection, and Monotonicity, $(\neg K)^2 = (\neg K)^{2n}$ for all $n \in \mathbb{N}$. We prove this fact in the Appendix (Lemma A.1).[15] This preliminary result implies that $U^{(\infty)} = U^{(n)}$ for all $n \geq 3$. That is, if a player is k^3-unaware of an event E (i.e., the chain of the lack of knowledge holds repeatedly three times), then she is indeed k^∞-unaware of E (i.e., this chain continues *ad infinitum*). Hence, as long as the notions of unawareness are derived

[15] Mathematically, this property is related to the notion of regularly open/closed sets in general topology (see, for example, Willard [33]) in the sense that K satisfies a part of the properties of the interior operator in a topological space.

from the lack of knowledge, we can restrict attention to $U^{(n)}$ with $n \in \{2, \infty\}$ and we can replace $U^{(\infty)}$ with $U^{(3)}$.[16] Note that $U^{(2)}$ and $U^{(\infty)} (= U^{(3)})$ are generally different (e.g., players i_1 and i_3 in Example 1).[17]

This observation leads to the following restatement of unawareness when knowledge satisfies Truth Axiom, Positive Introspection, and Monotonicity.

Proposition 1. *Fix any* $E \in \mathcal{D}$.

1. $U^{(\infty)}(E) = U^{(2)}(\neg K)(E)$. *Equivalently,* $A^{(\infty)}(E) = A^{(2)}(\neg K)(E)$.
2. $U^{(\infty)}(E) = \partial K(\neg K)(E) = \partial LK(E)$. *Also,* $U^{(\infty)}(E^c) = \partial KL(E)$.
3. $U^{(\infty)}(E) = LK(E) \setminus KLK(E)$. *Also,* $U^{(\infty)}(E^c) = LKL(E) \setminus KL(E)$.
4. $U^{(2)}(E) = \partial K(E) (\subseteq \partial(E))$. *Also,* $U^{(2)}(E^c) = \partial L(E) (\subseteq \partial(E))$.
5. $U^{(2)}(E) = LK(E) \setminus K(E)$. *Also,* $U^{(2)}(E^c) = L(E) \setminus KL(E)$.

The first three statements in Proposition 1 characterize k^∞-unawareness. The first result relates $U^{(\infty)}$ and $U^{(2)}$ in that a player is k^∞-unaware of an event E iff she is k^2-unaware of not knowing E.

The second and third statements characterize k^∞-unawareness by ignorance and possibility: a player is k^∞-unaware of an event E iff she does not know whether she knows that she does not know E (i.e., she is ignorant of knowing that she does not know E) iff she does not know whether it is possible that she knows E (i.e., she is ignorant of the possibility that she knows E). To restate further, a player is k^∞-unaware of an event E iff she considers it possible that she knows E but she does not know that it is possible that she knows E. At an interpretational level, the equivalence between the second and third statements lies in the fact that the notion of possibility is defined by the lack of knowledge (as well as in the given properties of knowledge (Truth Axiom, Positive Introspection, and Monotonicity)).

In the fourth and fifth statements, we characterize k^2-unawareness by ignorance and possibility: a player is k^2-unaware of an event E iff she does not know whether she knows E (i.e., she is ignorant of (not) knowing E). To restate further, she is k^2-unaware of E iff she considers it possible that she knows E but she does not know E. Also, (k^2-)unawareness implies ignorance: if a player is (k^2-)unaware of an event E then she is ignorant of E.

[16] If $\mathcal{D}$ is an algebra of sets, then we can restrict attention to $U^{(n)}$ with $n \in \{2, 3\}$ among any finite $n \geq 2$.

[17] Indeed, we will show in Proposition 4 that if $U^{(2)} = U^{(\infty)}$ then $U^{(2)}$ is degenerate in the sense that $U^{(2)}(\cdot) = (\neg K)(\Omega)$.

3.2 Characterization of Non-triviality

We say that $\mathcal{S} = \langle(\Omega, \mathcal{D}), (K, U)\rangle$ represents a *non-trivial form of unawareness* (or $\mathcal{S}$ is *non-trivial*) if there exists $E \in \mathcal{D}$ such that $U(E) \neq \emptyset$. We say that $\mathcal{S}$ is trivial otherwise. DLR [8, Theorem 1] show that any possibility correspondence model cannot capture unawareness if an unawareness operator satisfies Plausibility, KU Introspection, and AU Introspection. MR [24] show that if unawareness satisfies Symmetry ($U(E) = U(E^c)$) then $\mathcal{S}$ is trivial.

Given DLR's [8] negative result, the following two questions naturally arise. First, when does a state space model represent a non-trivial form of unawareness? Second, what properties have to be retained in order to represent a non-trivial form of unawareness in a state space model (DLR [8, p. 166])?

This subsection examines the above first question by providing a necessary and sufficient condition for an information structure to be non-trivial. Our characterization implies that $\mathcal{S}^{(2)} = \langle(\Omega, \mathcal{D}), (K, U^{(2)})\rangle$ is generically non-trivial even when it is induced from a reflexive and transitive possibility correspondence model, in the sense that $\mathcal{S}$ is non-trivial as long as it is not a partitional model.

Throughout this subsection, we fix an information structure $\mathcal{S}^{(n)} = \langle(\Omega, \mathcal{D}), (K, U^{(n)})\rangle$, where $n \in \{2, \infty\}$. Now, we characterize the non-triviality.

Proposition 2. *1. $U^{(2)}(E) \neq \emptyset$ iff $K(E) \in \mathcal{J} \setminus \mathcal{A}$ (i.e., $K(E) \subsetneq LK(E)$).*

2. $U^{(\infty)}(E) \neq \emptyset$ iff $LK(E) \in \mathcal{A} \setminus \mathcal{J}$ (i.e., $KLK(E) \subsetneq LK(E)$).

Corollary 1. *1. $\mathcal{S}^{(2)}$ is non-trivial iff $\mathcal{J} \setminus \mathcal{A} \neq \emptyset$ iff $\mathcal{A} \setminus \mathcal{J} \neq \emptyset$ iff $\mathcal{A} \triangle \mathcal{J} (:= (\mathcal{J} \setminus \mathcal{A}) \cup (\mathcal{A} \setminus \mathcal{J})) \neq \emptyset$.*

2. $\mathcal{S}^{(\infty)}$ is non-trivial iff $\{F \in \mathcal{J} \setminus \mathcal{A} \mid L(F) \in \mathcal{A} \setminus \mathcal{J}\} \neq \emptyset$.

We make four remarks. First, the triviality of a partitional information structure follows from $\mathcal{J} = \mathcal{A}$. Recall that Negative Introspection implies that $\mathcal{J}$ is closed under complementation.

Second, as an immediate implication of Corollary 1, the failure of Necessitation implies the non-triviality, although the non-triviality is rather degenerate.[18] This follows because $K(\Omega) \subsetneq \Omega = LK(\Omega)$.[19] In general, the unawareness operators satisfy $(\neg K)(\Omega) \subseteq U^{(\infty)}(E) \subseteq U^{(2)}(E)$ for all $E \in \mathcal{D}$. At any state $\omega \in (\neg K)(\Omega)$, an agent does not know anything and she is unaware of everything.

Third, consider $\mathcal{S}^{(2)}$. Since it is non-trivial iff $\mathcal{J}$ is not closed under complementation, it follows that any reflexive and transitive possibility correspondence model

[18] The converse is not true. Players i_1 and i_3 in Example 1 are such examples. We will shortly study the (non-)triviality of information structures satisfying Necessitation.

[19] Since $K(\neg K)(\Omega)$ is the smallest self-evident event, we have $K(\neg K)(\Omega) = \emptyset$, i.e., $\Omega = (\neg K)^2(\Omega) = LK(\Omega)$.

is non-trivial as long as it is not partitional. An underlying intuition is very simple: $\mathcal{S}^{(2)}$ is non-trivial iff Negative Introspection is violated. Thus, any (properly) non-partitional model of knowledge represents a non-trivial form of unawareness.

Fourth, consider an information structure $\mathcal{S}^{(n)}$ which satisfies Necessitation. Noting that $K(E) = \emptyset$ implies $LK(E) = \emptyset$, the fact that an agent is (k^n-)unaware of an event $E \in \mathcal{D}$ implies that she knows E at some state ω and she does not know E at another state ω'. If she does not know E at any state, then she knows that she does not know E at any state (because the event that she does not know E becomes a tautology), and thus she is not unaware of E at any state.

3.3 Properties of Unawareness

We keep considering a single player's information structure $\mathcal{S}^{(n)} = \langle(\Omega, \mathcal{D}), (K, U^{(n)})\rangle$ with $n \in \{2, \infty\}$. Here, we first examine "positive" properties of $U^{(n)}$. Next, we study "negative" properties under which unawareness becomes rather degenerate.

First, by definition, any $\mathcal{S}^{(n)}$ satisfies Plausibility. Also, $\mathcal{S}^{(\infty)}$ satisfies Strong Plausibility (HMS [16, 17] and Schipper [31]): $U^{(\infty)}(E) \subseteq \bigcap_{r\in\mathbb{N}}(\neg K)^r(E)$ with equality.[20] Note that the statement that $U^{(n)}(\cdot) \subseteq \partial(\cdot)$ (i.e., unawareness implies ignorance) could also be regarded as a plausibility condition.

Second, any $\mathcal{S}^{(n)}$ satisfies KU Introspection. DLR [8, Footnote 10] note that any pair (K, U) which satisfies Truth Axiom, Monotonicity, and Plausibility also satisfies KU Introspection. Now, we provide other properties that any $\mathcal{S}^{(n)}$ satisfies.

Proposition 3. *Any $\mathcal{S}^{(n)}$ satisfies the following.*

1. *Weak Necessitation: $A^{(n)}(E) \subseteq K(\Omega)$. Also, $A^{(n)}U^{(n)}(E) = K(\Omega)$.*
2. *Reverse AU Introspection: $U^{(n)}U^{(n)}(E) \subseteq U^{(n)}(E)$. Also, $U^{(n)}U^{(n)}U^{(n)}(E) = U^{(n)}U^{(n)}(E)$.*
3. *JU Introspection: $U^{(n)}(E) = \partial U^{(n)}(E)$. Equivalently, $A^{(n)}(E) = JA^{(n)}(E)$.*
4. *Weak A-Negative Introspection: $(\neg K)(E) \cap A^{(2)}(E) = K(\neg K)(E)$.*
5. *AK Self-Reflection: $A^{(n)}(E) = A^{(n)}K(E)$. Equivalently, $U^{(n)}(E) = U^{(n)}K(E)$.*
6. *A-Introspection: $A^{(n)}(E) = KA^{(n)}(E)$. Equivalently, $U^{(n)}(E) = LU^{(n)}(E)$.*
7. *Weak AA Self-Reflection: $A^{(n)}(E) \subseteq A^{(n)}A^{(n)}(E)$ with equality if $n = 2$. Also, $A^{(n)}A^{(n)}(E) = A^{(n)}A^{(n)}A^{(n)}(E)$.*

If K satisfies Finite Conjunction, then the following is also true.

[20] We also use other terminologies coined by HMS [16, 17] and Schipper [31] (specifically, Properties 5, 6, and 8) postulated in Proposition 3.

8. *A-Weak Conjunction:* $\bigcap_{E\in\mathcal{E}} A^{(n)}(E) \subseteq A^{(n)}(\bigcap \mathcal{E})$*, where* $\mathcal{E}$ *is a non-empty finite subset of* $\mathcal{D}$*.*

Weak Necessitation (Property 1) is coined by DLR [8]. Property 2 is the "converse" of AU Introspection. We note that Plausibility, KU Introspection, and Weak Necessitation imply Reverse AU Introspection. Similarly, the ignorance operator also satisfies $\partial\partial(\cdot) \subseteq \partial(\cdot)$.

JU Introspection (Property 3) states that if a player is unaware of an event then she is *ignorant* of being unaware of it. Reverse AU Introspection is also seen as a consequence of JU Introspection since $U^{(n)}(\cdot) \subseteq \partial(\cdot)$ (see Proposition 1).

Weak A-Negative Introspection (Property 4) is proposed by Li [21]. If $n = \infty$, then this is equivalent to Weak Negative Introspection (Fagin and Halpern [11]) for $n = 2$: $(\neg K)(E) \cap A^{(2)}(\neg K)(E) = K(\neg K)(E)$, since $A^{(\infty)}(E) = A^{(2)}(\neg K)(E)$.

Weak A-Negative Introspection for $n = \infty$ (equivalently, Weak Negative Introspection for $n = 2$), however, may not hold (specifically, the "$\subseteq$" part may fail). That is, there may exist a state at which the following hold: an agent does not know an event E and she is (k^2-)aware that she does not know E but she does not know that she does not know E. Indeed, observe that Weak A-Negative Introspection is equivalent to $(\neg K)(\cdot) \cap (\neg K)^2(\cdot) \subseteq U^{(n)}(\cdot)$, and hence this may not hold for $n = \infty$.[21] As a particular example, consider Example 1. We have $K_{i_1}(\neg K_{i_1})(\{\omega_1, \omega_2\}) = \emptyset$ while $(\neg K_{i_1})(\{\omega_1, \omega_2\}) \cap A_{i_1}^{(\infty)}(\{\omega_1, \omega_2\}) = \{\omega_3\}$. Moreover, $(\neg K_{i_1})(\{\omega_1, \omega_2\}) \cap A_{i_1}^{(\infty)}(\neg K_{i_1})(\{\omega_1, \omega_2\}) = \{\omega_3\}$.

Properties 5,6, and 8 are proposed by MR [24, 25]. AK Self-Reflection (Property 5) is equivalently stated as $U^{(n)}(\cdot) = U^{(n)}K(\cdot)$: an agent is unaware of E iff she is unaware of knowing E. On the other hand, A-Introspection (Property 6) is equivalently stated as $U^{(n)} = LU^{(n)}$: an agent is unaware iff she considers it possible that she is unaware. It is equivalent to $U^{(n)} = (\neg K)A^{(n)}$, and hence an agent is unaware iff she does not know that she is aware.

Note that it is not necessarily true that $U^{(n)}(\cdot) = U^{(n)}L(\cdot)$: an agent is unaware of an event E iff she is unaware of the possibility of E.[22] Consider Example 1: $U_{i_1}^{(2)}(\{\omega_1, \omega_2\}) = \{\omega_3\}$ while $U_{i_1}^{(2)}L_{i_1}(\{\omega_1, \omega_2\}) = \emptyset$.

Property 7 (Weak AA Self-Reflection) is based on AA Self-Reflection (MR [24, 25]): $A^{(n)}(E) = A^{(n)}A^{(n)}(E)$. While AA Self-Reflection holds when $n = 2$, only the weak form (Property 6) is true when $n = \infty$. For instance, in Example 1, we have $A_{i_1}^{(\infty)}(\{\omega_1\}) = \{\omega_1, \omega_2\}$ while $A_{i_1}^{(\infty)}A_{i_1}^{(\infty)}(\{\omega_1\}) = \Omega$.

Consider Property 8 (A-Weak Conjunction). We have two remarks. First, it may not necessarily hold with equality. In Example 1, we have $A_{i_1}^{(n)}(\{\omega_1, \omega_3\}) \cap$

[21] Given an information structure $\langle(\Omega, \mathcal{D}), (K, U)\rangle$, one set of axioms that induces $U = U^{(2)}$ trivially turns out to be Plausibility and Weak A-Negative Introspection. It would be an interesting question to ask whether there are other "non-trivial" combinations of axioms that yield $U = U^{(2)}$ given a pair (K, U).

[22] We have $U^{(2)}L(E) = U^{(2)}(\neg K)(E^c) = U^{(\infty)}(E^c)$ and $U^{(\infty)}L(E) = (\neg K)^3(E^c) \cap (\neg K)^4(E^c) = (\neg K)^3(E^c) \cap (\neg K)^2(E^c) = U^{(\infty)}(E^c)$.

$A_{i_1}^{(n)}(\{\omega_2, \omega_3\}) = \{\omega_1, \omega_2\} \subsetneq \Omega = A_{i_1}^{(n)}(\{\omega_3\})$ for $n \in \{2, \infty\}$. Second, A-Weak Conjunction holds for any non-empty collection of events $\mathcal{E}$ with respect to $A^{(2)}$ in any information structure satisfying Non-empty Conjunction. We, however, leave it an open question whether this extends to the case of $n = \infty$.

Let us go back to the question raised by DLR [8, p. 166], which of their three axioms is to be retained in a possibility correspondence model so as to capture a non-trivial form of unawareness. One implication of Proposition 3 is that any information structure satisfies Plausibility (by definition), KU Introspection, Reverse AU Introspection, and JU Introspection (instead of AU Introspection).

Now, we turn to examining other properties which lead to a degenerate form of unawareness. Especially, these properties lead to trivial unawareness under Necessitation.

Proposition 4. *Let $\mathcal{S}^{(n)}$ be an information structure.*

1. *Let $n = 2$. (a)-(g) are all equivalent to $U^{(2)}(\cdot) = (\neg K)(\Omega)(= L(\emptyset))$.*
2. *Let $n = \infty$. (f)-(i) are all equivalent to $U^{(\infty)}(\cdot) = (\neg K)(\Omega)(= L(\emptyset))$.*

(a) *Subjective Negative Introspection: $K(\Omega) \setminus K(E) \subseteq K(K(\Omega) \setminus K(E))$.*

(b) *If $E \in \mathcal{J}$ then $K(\Omega) \setminus E \in \mathcal{J}$.*

(c) *Negative Non-Introspection: $(\neg K)(E) \cap (\neg K)^2(E) \subseteq (\neg K)^3(E)$.*

(d) *$U^{(2)} = U^{(\infty)}$.*

(e) *Symmetry of $U^{(2)}$: $U^{(2)}(E) = U^{(2)}(E^c)$.*

(f) *AU introspection of $U^{(n)}$.*

(g) *Monotonicity of $U^{(n)}$: if $E \subseteq F$ then $U^{(n)}(E) \subseteq U^{(n)}(F)$.*

(h) *Monotonicity of $A^{(n)}$: if $E \subseteq F$ then $A^{(n)}(E) \subseteq A^{(n)}(F)$ (i.e., $U^{(n)}(F) \subseteq U^{(n)}(E)$).*

(i) *AA Self Reflection of $A^{(\infty)}$: $A^{(\infty)}(E) = A^{(\infty)}A^{(\infty)}(E)$.*

We make the following four remarks. First, while (a) states that $K(\Omega) \setminus K(E)$ is self-evident, (b) states that if E is self-evident then so is $K(\Omega) \setminus E$. The equivalence between (a) and (f) (i.e., AU Introspection) is closely related to CEL [6] in the sense that Subjective Negative Introspection reduces to Negative Introspection under Necessitation.

Second, the terminology, Negative Non-Introspection, is from Schipper [31]. It is clearly equivalent to (d). This part says that unawareness is degenerate when $U^{(2)} = U^{(\infty)}$.

Third, MR [24, Theorem] show the triviality of k^2-unawareness under Symmetry. While Symmetry of $U^{(2)}$ yields a rather degenerate form of unawareness, Symmetry of $U^{(\infty)}$ does not necessarily imply this property (e.g., player i_1 in Example 1).

Fourth, if $U^{(2)}(\cdot) = (\neg K)(\Omega)$ then we have $U^{(2)} = U^{(\infty)}$. However, $U^{(\infty)}(\cdot) = (\neg K)(\Omega)$ does not necessarily imply $U^{(2)} = U^{(\infty)}$ (e.g., player i_3 in Example 1).

4 Further Properties of Unawareness

4.1 Knowledge of Self-awareness

We ask the *knowledge of self-unawareness*, i.e., we ask whether there exists a state in which an agent knows that she is unaware of "something" even though KU Introspection requires that she do not know that she is unaware of any *particular* event.[23]

Throughout this subsection, fix $\mathcal{S}^{(n)} = \langle(\Omega, \mathcal{D}), (K, U^{(n)})\rangle$ with $n \in \{2, \infty\}$. We denote the event that an agent is unaware of something by $\overline{U}^{(n)} := \{\omega \in \Omega \mid \omega \in U^{(n)}(E) \text{ for some } E \in \mathcal{D}\} = \bigcup_{E\in\mathcal{D}} U^{(n)}(E)$. Note that $\overline{U}^{(n)}$ is a well-defined event (i.e., $\overline{U}^{(n)} \in \mathcal{D}$) and that $\mathcal{S}^{(n)}$ is non-trivial iff $\overline{U}^{(n)} \neq \emptyset$.

Proposition 5. *1. Assume Finite Conjunction on K. If $\mathcal{D}$ is finite, then $K(\overline{U}^{(n)}) = \emptyset$ and $A^{(n)}(\overline{U}^{(n)}) = K(\Omega)$ (i.e., $U^{(n)}(\overline{U}^{(n)}) = (\neg K)(\Omega)$). It is possible that $K(\overline{U}^{(n)}) \neq \emptyset$ when $\mathcal{D}$ is infinite.*

2. $A(\overline{U}^{(n)}) \neq \emptyset$ provided that $K(\Omega) \neq \emptyset$. Also, $U^{(n)}(\overline{U}^{(n)}) \subseteq \overline{U}^{(n)}$.

3. If K satisfies Arbitrary Conjunction, then $\overline{U}^{(n)} = L(\overline{U}^{(n)})$.

The first part of Proposition 5 states that, for any information structure satisfying Finite Conjunction, if the domain is finite then an agent never knows that she is unaware of something (i.e., $K(\overline{U}^{(n)}) = \emptyset$). Thus, whenever she knows something, she infers that she never knows that she is unaware of something: she is aware that she is unaware of something (i.e., $K(\Omega) = K(\neg K)(\overline{U}^{(n)}) = A^{(n)}(\overline{U}^{(n)})$). On the other hand, it is possible that an agent knows that she is unaware of something, if a given domain is infinite, even though she never knows that she is unaware of any particular event.

The second part states the following. Suppose that an agent's knowledge is not degenerate in the sense that $K(E) \neq \emptyset$ for some $E \in \mathcal{D}$. Then, there is always a state at which the agent is aware of being unaware of something.

The third part implies that an agent is unaware of something iff she considers it possible that she is unaware of something in any information structure satisfying Arbitrary Conjunction. Note that the statement is true for any information structure

[23] See also Schipper [31, Section 3.5] and the references therein for syntactic approaches to self-awareness.

satisfying Non-empty Conjunction as long as $\overline{U}^{(n)} \neq \emptyset$. If a given domain is finite then the statement is clearly true for any information structure satisfying Finite Conjunction.

4.2 Non-monotonicity of Unawareness in Knowledge

Proposition 2 and Corollary 1 show that the non-triviality of unawareness hinges on the qualitative feature of knowledge (e.g., whether the lack of knowledge is self-evident when unawareness is defined by the two levels of the lack of knowledge). Thus, the non-triviality is not related to knowledgeability.

Here, we take a further look at the non-monotonicity of unawareness in knowledgeability through examples. An underlying intuition is that, while an increase in knowledge enhances awareness through knowledge itself, a decrease in knowledge also enhances awareness through the knowledge of the lack of knowledge. In an extreme case, an agent with her self-evident collection $\mathcal{J} = \{\emptyset, \Omega\}$ is not unaware of any event. This is because she always knows that she does not any non-tautological event.

Now, we examine three cases in which knowledge and unawareness exhibit non-monotonicity. In order to keep the discussion simple, we use Example 1. First, we consider the "monotonicity of awareness in knowledge." One individual is at least as knowledgeable as another at a certain state, and the better informed individual is aware of any event, at that state, of which the less informed individual is aware at that state. Consider players i_1 and i_4 in Example 1. Player i_1 is at least as knowledgeable as i_4 at any state, and i_1 is aware of any event of which i_4 is aware at any state.

The second case, on the other hand, exhibits "monotonicity of unawareness in knowledge." One agent is at least as knowledgeable as another at a given state, and the better informed agent is unaware, at that state, of any event of which the less informed agent is unaware at that state. Again, consider player i_1 in Example 1 and player whose knowledge is defined by $\{\emptyset, \Omega\}$. As Corollary 1 implies, the latter player is aware of every event at each state. In this example the less informed agent knows her own ignorance, which leads to more awareness.

In the third case, we consider two individuals who are equally knowledgeable at a given state, where one is unaware of an event at that state while the other is aware of it at the state. For example, consider players i_2 and i_3 in Example 1. At state $\omega \in \{\omega_2, \omega_3\}$, they are equally knowledgeable. Consider state ω_2. Player i_3 is not unaware of any event $E \in \{\emptyset, \{\omega_2\}, \{\omega_3\}, \{\omega_2, \omega_3\}, \Omega\}$ while player i_2 is unaware of every event. Consider ω_3. Player i_3 is not unaware of any event $E \in \{\emptyset, \{\omega_2\}, \{\omega_3\}, \{\omega_2, \omega_3\}, \Omega\}$ while player i_2 is aware of every event.

Finally, we have two remarks. First, we can view such comparison of players' knowledge and unawareness as one player's knowledge and unawareness over time. To that end, let a state space be given by $\Omega = \{\omega_1, \omega_2, \omega_3\}$ as in Example 1. Denote an individual i's knowledge at time t by $\mathcal{J}_{i(t)}$. Specifically, we let $\mathcal{J}_{i(0)} = \mathcal{J}_{i_4}$, $\mathcal{J}_{i(1)} = \mathcal{J}_{i_1}$, and $\mathcal{J}_{i(2)} = \{\emptyset, \Omega\}$. At time 1, getting more information causes individual i to get aware of some event at each realized state. At time 2, on the other hand, she "forgets" some events, and this may make her aware of some events at some states.

Second, the entire discussion also applies to common knowledge. It is possible that if some event is not commonly known then it is commonly known that this is not common knowledge. When each individual receives some events, on the contrary, it may become possible that it is not common knowledge that this is not common knowledge.

4.3 Possible Forms of Monotonicity of Unawareness in Knowledge

We examine possible forms of monotonicity of unawareness in knowledgeability. The key observation is monotonicity of the knowledge and ignorance operators in knowledgeability. Thus, if j is at least as knowledgeable as i, then KU Introspection of $(K_j, U_j^{(n)})$ implies that $K_i U_j^{(n)}(E) = \emptyset$.

Ignorance is "decreasing" in knowledge because ignorance of an event E is expressed in terms of the lack of knowledge of E and its negation E^c. That is, for any event E, if j is ignorant of E then so is i, provided that j is at least as knowledgeable as i. Monotonicity of these operators on knowledgeability implies the following.

Proposition 6. *Let j be at least as knowledgeable as i. Fix $n \in \{2, \infty\}$ and $E \in \mathcal{D}$.*

1. (a) $\partial_j(K_i E) \subseteq U_i^{(2)}(E)$. *Equivalently,* $A_i^{(2)}(E) \subseteq J_j(K_i E)$.

   (b) $\partial_j(L_i K_i E) \subseteq U_i^{(\infty)}(E)$. *Equivalently,* $A_i^{(\infty)}(E) \subseteq J_j(L_i K_i E)$.

   (c) $\partial_j U_i^{(n)}(E) \subseteq U_i^{(n)}(E)$ *and* $U_j^{(n)}(E) \subseteq \partial_i U_j^{(n)}(E)$.

2. (a) $U_j^{(2)}(E) \subseteq \partial_i(K_j E)$. *Equivalently,* $J_i(K_j E) \subseteq A_j^{(2)}(E)$.

   (b) $U_j^{(\infty)}(E) \subseteq \partial_i(L_j K_j E)$. *Equivalently,* $J_i(L_j K_j E) \subseteq A_j^{(\infty)}(E)$.

3. $A_i^{(n)}(E) = K_j A_i^{(n)}(E) = A_i^{(n)} K_j K_i(E)$. *Also,* $U_i^{(n)}(E) = L_j U_i^{(n)}(E) = U_i^{(n)} K_j K_i(E)$.

4. $A_i^{(n)}(E) \subseteq A_j^{(n)} A_i^{(n)}(E)$ *and* $U_j^{(n)} U_i^{(n)}(E) \subseteq U_i^{(n)}(E)$.

Suppose that an individual j is at least as knowledgeable as i. The first two statements are comparative statics of unawareness with respect to ignorance. First, if j is ignorant of i knowing E, then i is (k^2-)unaware of E. Second, on the contrary, if j is (k^2-)unaware of E, then i is ignorant of j knowing E.

The third statement says that i's awareness of an event is self-evident to j. Likewise, we show in the fourth statement that i's awareness of an event E implies j's awareness of i's awareness of E. We also show that if j is unaware of i's unawareness of an event E then i is indeed unaware of E.

While we demonstrated possible forms of monotonicity of unawareness, it is, however, to be noted that the fact that $K_i(\cdot) \subseteq K_j(\cdot)$ does not necessarily imply that $U_j(\cdot) \subseteq U_i(\cdot)$ (Example 1 with $(i, j) = (i_1, i_3)$).

Finally, the first part of the next proposition compares a player's unawareness at different states, while the second part compares different players' unawareness at a given state.

Proposition 7. *Fix $n \in \{2, \infty\}$.*

1. *Suppose that state ω' is as informative as ω in the sense that $\mathrm{IK}_i(\omega) \subseteq \mathrm{IK}_i(\omega')$. If $\omega' \in U_i^{(n)}(E)$ then $\omega \in U_i^{(n)}(E)$ (i.e., if $\omega \in A_i^{(n)}(E)$ then $\omega' \in A_i^{(n)}(E)$).*

2. *Suppose that j is at least as knowledgeable as i at ω. If $\omega \in U_j^{(n)}(E)$, then $\omega \notin K_i U_j^{(n)}(E)$. If $\omega \in A_i^{(n)}(E)$, then $\omega \in K_j A_i^{(n)}(E)$.*

The first part of Proposition 7 says that if i is unaware of an event E at ω' and if ω' ranks no lower than ω in her informativeness relation (i.e. she knows at ω' any event that she knows at ω), then she is also unaware of E at ω. On the other hand, with regard to the relation, at least as knowledgeable as at a state, if an individual j is at least as knowledgeable as an individual i at a state ω and if j is unaware of an event E there, then i does not know that j is unaware of E there. Interestingly, it is not always true that if an individual j is at least as knowledgeable as an individual i at a state ω and if j is unaware of an event E there, then i is unaware of E there.

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A Appendix

A.1 Section 3.1

In order to prove Proposition 1, we consider the following preliminary lemma.

Lemma A.1. *Fix* $E \in \mathcal{D}$.

1. $K(E) \subseteq LK(E) = (\neg K)^2(E) = (\neg K)^{2n}(E) \subseteq (\neg K)(E^c)(= L(E))$.
2. $((\neg L)(E) =)K(E^c) \subseteq (\neg K)^{2n+1}(E) = (\neg K)^3(E) \subseteq (\neg K)(E)$.

Proof of Lemma A.1. It is enough to show (1). First, Truth Axiom and Monotonicity imply that $K(E) \subseteq LK(E) \subseteq L(E)$. Second, by definition, we have $L(E) = (\neg K)(E^c)$ and $LK(E) = (\neg K)^2(E)$. Now, it suffices to show $(\neg K)^2(E) = (\neg K)^4(E)$, i.e., $LK(E) = LKLK(E)$. Since Truth Axiom implies $K(E) \subseteq LK(E)$, Positive Introspection and Monotonicity imply $K(E) \subseteq KK(E) \subseteq KLK(E)$. By Monotonicity, we get $LK(E) \subseteq LKLK(E)$. Conversely, Truth Axiom implies $KLK(E) \subseteq LK(E)$. Monotonicity and Positive Introspection imply $LKLK(E) \subseteq LLK(E) \subseteq LK(E)$. □

Proof of Proposition 1. Part 1. By Lemma A.1, we have $U^{(\infty)}(E) = U^{(3)}(E) = (\neg K)^2(E) \cap (\neg K)^3(E) = U^{(2)}(\neg K)(E)$. Equivalently, $A^{(\infty)}(E) = A^{(3)}(E) = K(\neg K)(E) \cup K(\neg K)^2(E) = A^{(2)}(\neg K)(E)$.

Part 4. First, we have $U^{(2)}(E) = (\neg K)(E) \cap (\neg K)^2(E) = (\neg K)(KE) \cap (\neg K)(\neg KE) = \partial(\neg K)(E) = \partial K(E)$. Second, $\partial KE = (\neg K)KE \cap (\neg K)(\neg K)(E) = (\neg K)E \cap LKE \subseteq (\neg K)E \cap LE = (\neg K)E \cap (\neg K)(E^c) = \partial E$. Third, we have $U^{(2)}(E^c) = \partial K(E^c) = \partial(\neg K)(E^c) = \partial L(E)$. Also, $\partial L(E) = \partial K(E^c) \subseteq \partial(E^c) = \partial(E)$.

Part 5. We have $U^{(2)}(E) = LK(E) \cap (\neg K)K(E) = LK(E) \setminus K(E)$. Also, $U^{(2)}(E^c) = (\neg K)(\neg K)(E^c) \cap (\neg K)K(E^c) = (\neg K)L(E) \cap L(E) = L(E) \setminus KL(E)$.

Part 2. It follows that $U^{(\infty)}(E) = U^{(2)}(\neg K)(E) = \partial K(\neg K)(E) = \partial(\neg K)^2(E) = \partial LK(E)$. Also, $U^{(\infty)}(E^c) = \partial K(\neg K)(E^c) = \partial KL(E)$.

Part 3. We obtain $U^{(\infty)}(E) = \partial LK(E) = (\neg K)LK(E) \cap (\neg K)(\neg L)K(E) = LLK(E) \setminus KLK(E) = LK(E) \setminus KLK(E)$. Likewise, $U^{(\infty)}(E^c) = \partial KL(E) = (\neg K)KL(E) \cap (\neg K)(\neg K)L(E) = (\neg K)L(E) \cap LKL(E) = LKL(E) \setminus KL(E)$. □

As a remark, we show below further properties of unawareness.

Proposition A.1. *Fix* $E \in \mathcal{D}$.

1. $U^{(2)}(E) \cap U^{(2)}(E^c) \subseteq U^{(\infty)}(E)$.
2. $U^{(2)}(E) \cup U^{(2)}(E^c) \subseteq U^{(2)}J(E)$.

Proof of Proposition A.1. 1. We have the following:

$$\begin{aligned} U^{(2)}(E) \cap U^{(2)}(E^c) &= (\neg K)(E) \cap (\neg K)^2(E) \cap (\neg K)(E^c) \cap (\neg K)^2(E^c) \\ &= (\neg K)(E) \cap (\neg K)^2(E) \cap L(E) \cap (\neg K)L(E) \\ &\subseteq (\neg K)(E) \cap (\neg K)^2(E) \cap (\neg K)LK(E) \\ &= (\neg K)(E) \cap (\neg K)^2(E) \cap (\neg K)^3(E) = U^{(\infty)}(E). \end{aligned}$$

2. Noting that $KJ(E) = J(E)$, we have the following:

$$\begin{aligned} U^{(2)}J(E) &= (\neg K)J(E) \cap (\neg K)^2 J(E) = \partial(E) \cap (\neg K)\partial(E) \\ &= \partial(E) \cap (\neg K)((\neg K)(E) \cap (\neg K)(E^c)) \\ &\supseteq \partial(E) \cap \neg(K(\neg K)(E) \cap K(\neg K)(E^c)) \\ &= \partial(E) \cap ((\neg K)^2(E) \cup (\neg K)^2(E^c)) \\ &= (\partial(E) \cap (\neg K)^2(E)) \cup (\partial(E) \cap (\neg K)^2(E^c)) \\ &= (\partial(E) \cap (\neg K)^2(E)) \cup (\partial(E^c) \cap (\neg K)^2(E^c)). \end{aligned}$$

Now, since Lemma A.1 implies that

$$\partial(E) \cap (\neg K)^2(E) = (\neg K)(E) \cap (\neg K)^2(E) \cap (\neg K)(E^c) = U^{(2)}(E),$$

we obtain $U^{(2)}J(E) \supseteq U^{(2)}(E) \cup U^{(2)}(E^c)$. The equality holds when K satisfies Finite Conjunction.

□

A.2 Section 3.2

Proof of Proposition 2. 1. Suppose that there is $E \in \mathcal{D}$ such that $U^{(2)}(E) \neq \emptyset$. Since $U^{(2)}(E) = (\neg K)(E) \cap LK(E)$, we must have $K(E) \neq LK(E)$. Then we have $K(E) \in \mathcal{J} \setminus \mathcal{A}$.

Conversely, suppose that there is $E \in \mathcal{D}$ with $K(E) \in \mathcal{J} \setminus \mathcal{A}$. Then, there is $\omega \in LK(E) \setminus K(E)$. That is, $U^{(2)}(E) = LK(E) \setminus K(E) \neq \emptyset$.

2. Suppose that $\emptyset \neq U^{(\infty)}(E)$ for some $E \in \mathcal{D}$. Since $U^{(\infty)}(E) = LK(E) \setminus KLK(E)$, we have $KLK(E) \neq LK(E)$. We obtain $LK(E) \in \mathcal{A} \setminus \mathcal{J}$.

Conversely, suppose that there is $E \in \mathcal{D}$ such that $LK(E) \in \mathcal{A} \setminus \mathcal{J}$. Then, we have $KLK(E) \subsetneq LK(E)$. That is, $U^{(\infty)}(E) = LK(E) \setminus KLK(E) \neq \emptyset$.

□

Proof of Corollary 1. 1. If $\mathcal{S}^{(2)}$ is non-trivial, then $U^{(2)}(E) \neq \emptyset$ for some $E \in \mathcal{D}$. It follows from Proposition 2 that $F := K(E) \in \mathcal{J} \setminus \mathcal{A}$, i.e., $\mathcal{J} \setminus \mathcal{A} \neq \emptyset$. Conversely, if $\mathcal{J} \setminus \mathcal{A} \neq \emptyset$, then there is $K(E) = E \in \mathcal{J} \setminus \mathcal{A}$, and hence $U^{(2)}(E) \neq \emptyset$, i.e., $\mathcal{S}^{(2)}$ is non-trivial. The rest follows because $E \in \mathcal{J} \setminus \mathcal{A}$ iff $E^c \in \mathcal{A} \setminus \mathcal{J}$.

2. If $\mathcal{S}^{(\infty)}$ is non-trivial, then $U^{(\infty)}(E) \neq \emptyset$ for some $E \in \mathcal{D}$. It follows from Proposition 2 that $F := K(E) \in \mathcal{J} \setminus \mathcal{A}$ such that $L(F) = LK(E) \in \mathcal{A} \setminus \mathcal{J}$. Hence, $\{E \in \mathcal{J} \setminus \mathcal{A} \mid L(E) \in \mathcal{A} \setminus \mathcal{J}\} \neq \emptyset$. Conversely, if $\{E \in \mathcal{J} \setminus \mathcal{A} \mid L(E) \in \mathcal{A} \setminus \mathcal{J}\} \neq \emptyset$, then there is $K(E) = E \in \mathcal{J} \setminus \mathcal{A}$ such that $L(E) = LK(E) \in \mathcal{A} \setminus \mathcal{J}$, and hence $U^{(\infty)}(E) \neq \emptyset$, i.e., $\mathcal{S}^{(\infty)}$ is non-trivial.

□

A.3 Section 3.3

Before we show Proposition 3, we prove an intermediate result, which will be used for proving (the last statement of) Proposition 3 (and Proposition 5).

Lemma A.2. *Let $\mathcal{S} = \langle (\Omega, \mathcal{D}), (K, U) \rangle$ be an information structure satisfying Finite Conjunction. For any $E, F \in \mathcal{D}$, we have $K(E \cup F) \cup L(F) \subseteq KE \cup LF$.*

Proof of Lemma A.2. First, Finite Conjunction implies that $K(E \cup F) \cap K(F^c) = K((E \cup F) \cap F^c) = K(E \cap F^c) = K(E) \cap K(F^c) \subseteq KE$. Then, we get $K(E \cup F) \cup LF = (K(E \cup F) \cap K(F^c)) \cup LF \subseteq KE \cup LF$. □

Proof of Proposition 3. Fix $E \in \mathcal{D}$. Since $A^{(n)}(E) = \bigcup_{r=1}^{n} K(\neg K)^{r-1}(E)$ is self-evident, A-Introspection ($A^n(E) = KA^{(n)}(E)$) clearly holds.

1. First, since Monotonicity implies that $K(F) \subseteq K(\Omega)$ for any $F \in \mathcal{D}$, it is clear that $A^{(n)}(E) \subseteq K(\Omega)$. Second, it follows from KU Introspection that $A^{(n)}U^{(n)}(E) \supseteq K(\neg K)U^{(n)}(E) = K(\Omega)$. Hence, $A^{(n)}U^{(n)}(E) = K(\Omega)$.

2. First, we have

$$(\neg K)(\Omega) \subseteq U^{(n)}U^{(n)}(E) \subseteq (\neg K)U^{(n)}(E) \cap (\neg K)^2 U^{(n)}(E) = (\neg K)(\Omega) \subseteq U^{(n)}(E),$$

   where the first set-inclusion is from Weak Necessitation, the second set-inclusion is from Plausibility, the next equality is from KU Introspection, and the last set-inclusion is from Weak Necessitation.[A.1] Second, since $U^{(n)}U^{(n)}(F) = (\neg K)(\Omega)$ for all $F \in \mathcal{D}$, we have $U^{(n)}U^{(n)}U^{(n)}(E) = U^{(n)}U^{(n)}(E)$.

3. JU Introspection follows from KU Introspection and A-Introspection: $\partial U^{(n)}(E) = (\neg K)U^{(n)}(E) \cap (\neg K)A^{(n)}(E) = U^{(n)}(E)$.

4. Weak A-Negative Introspection simply follows from the definition of $A^{(2)}$.

5. It follows from Positive Introspection that $A^{(n)}(KE) = \bigcup_{r=1}^{n} K(\neg K)^{r-1}KE = \bigcup_{r=1}^{n} K(\neg K)^{r-1}(E) = A^{(n)}(E)$.

6. We have already shown A-Introspection.

[A.1] Alternatively, it follows from $U^{(n)}(\cdot) \subseteq \partial(\cdot)$ (Proposition 1) and JU Introspection that $U^{(n)}U^{(n)}(E) \subseteq \partial U^{(n)}(E) = U^{(n)}(E)$.

7. First, let $n = 2$. It follows from A-Introspection and KU Introspection that $U^{(2)}A^{(2)}(E) = (\neg K)A^{(2)}(E) \cap (\neg K)(\neg K)A^{(2)}(E) = U^{(2)}(E) \cap (\neg K)U^{(2)}(E) = U^{(2)}(E)$. Then, we get $A^{(2)}(E) = A^{(2)}A^{(2)}(E)$.

   Next, let $n = \infty$. It also follows from A-Introspection that $U^{(\infty)}A^{(\infty)}(E) \subseteq (\neg K)A^{(\infty)}(E) = U^{(\infty)}(E)$. We get $A^{(\infty)}(E) \subseteq A^{(\infty)}A^{(\infty)}(E)$. Since $A^{(\infty)}A^{(\infty)}(E) = K(\Omega)$, we have $A^{(\infty)}A^{(\infty)}A^{(\infty)}(E) = A^{(\infty)}A^{(\infty)}(E)$.

8. We show that $U^{(n)}(\bigcap \mathcal{E}) \subseteq \bigcup_{E\in\mathcal{E}} U^{(n)}(E)$, where $\mathcal{E}$ is a non-empty finite subset of $\mathcal{D}$. For $n = 2$, it follows from Finite Conjunction (and Monotonicity) that

$$\begin{aligned} U^{(2)}(\bigcap \mathcal{E}) = (\neg K)(\bigcap \mathcal{E}) \cap (\neg K)^2(\bigcap \mathcal{E}) &= \bigcup_{E\in\mathcal{E}} (\neg K)(E) \cap (\neg K)^2(\bigcap \mathcal{E}) \\ &\subseteq \bigcup_{E\in\mathcal{E}} (\neg K)(E) \cap \bigcap_{E\in\mathcal{E}} (\neg K)^2(E) \\ &\subseteq \bigcup_{E\in\mathcal{E}} ((\neg K)(E) \cap (\neg K)^2(E)) = \bigcup_{E\in\mathcal{E}} U^{(2)}(E). \end{aligned}$$

   Note that the same proof clearly works for any non-empty (countable) set $\mathcal{E}$ if $\mathcal{S}$ satisfies Non-empty (Countable) Conjunction.

   Next, consider $n = \infty$. We show:

$$\begin{aligned} U^{(\infty)}(\bigcap \mathcal{E}) &= (\neg K)^2(\bigcap \mathcal{E}) \cap (\neg K)^3(\bigcap \mathcal{E}) \\ &\subseteq \bigcap_{E\in\mathcal{E}} (\neg K)^2(E) \cap (\neg K)^2(\bigcup_{E\in\mathcal{E}} (\neg K)(E)) \\ &\subseteq \bigcap_{E\in\mathcal{E}} (\neg K)^2(E) \cap \bigcup_{E\in\mathcal{E}} (\neg K)^3(E) \\ &\subseteq \bigcup_{E\in\mathcal{E}} ((\neg K)^2(E) \cap (\neg K)^3(E)) = \bigcup_{E\in\mathcal{E}} U^{(\infty)}(E), \end{aligned}$$

   where the second line follows from Finite Conjunction (and Monotonicity). Thus, it suffices to show the following for a finite Λ, which implies the third line:

$$(\neg K)^2(\bigcup_{\lambda\in\Lambda} F_\lambda) \subseteq \bigcup_{\lambda\in\Lambda} (\neg K)^2(F_\lambda), \text{ where } F_\lambda = (\neg K)(E_\lambda).$$

   Without loss of generality, assume $\Lambda = \{1, 2\}$. It follows from Lemma A.2 that $K(F_1 \cup F_2) \subseteq K(F_1) \cup L(F_2) \subseteq LK(F_1) \cup F_2$, where note that $LF_\lambda = F_\lambda$ for each λ. Then, Monotonicity and Finite Conjunction (of K) imply that $LK(F_1 \cup F_2) \subseteq L(LK(F_1) \cup F_2) = LK(F_1) \cup F_2$. On the other hand, it follows from Lemma A.2 that $K(LK(F_1) \cup F_2) \subseteq K(F_2) \cup LLK(F_1) \subseteq LK(F_1) \cup LK(F_2)$, and hence Monotonicity implies $LK(LK(F_1) \cup F_2) \subseteq L(LK(F_1) \cup LK(F_2)) = LK(F_1) \cup LK(F_2)$. Now, recalling that $LKLK(\cdot) = LK(\cdot)$ (see Lemma A.1), we have

$$LK(F_1 \cup F_2) = LKLK(F_1 \cup F_2) \subseteq LK(LK(F_1) \cup F_2) \subseteq LK(F_1) \cup LK(F_2).$$

$\square$

Remark A.1. We have established A-Weak Conjunction in Proposition 3. Here, we provide counterexamples for other forms of Conjunction and Disjunction with regards to $A^{(n)}$ and $U^{(n)}$. Let i_1 be as in Example 1.

1. Consider $A^{(n)}(\bigcup_{\lambda\in\Lambda} E_\lambda) = \bigcup_{\lambda\in\Lambda} A^{(n)}(E_\lambda)$. We have $A^{(n)}_{i_1}(\{\omega_1\}) \cup A^{(n)}_{i_1}(\{\omega_2, \omega_3\}) = \{\omega_1, \omega_2\} \subsetneq \Omega = A^{(n)}_{i_1}(\Omega)$. We also have $A^{(n)}_{i_1}(\{\omega_2\}) \cup A^{(n)}_{i_1}(\{\omega_3\}) = \Omega \supsetneq \{\omega_1, \omega_2\} = A^{(n)}_{i_1}(\{\omega_2, \omega_3\})$.

2. Consider $U^{(n)}(\bigcup_{\lambda\in\Lambda} E_\lambda) = \bigcup_{\lambda\in\Lambda} U^{(n)}(E_\lambda)$. First, consider the "$\subseteq$" part. We have $U^{(n)}_{i_1}(\{\omega_1\}) \cup U^{(n)}_{i_1}(\{\omega_2\}) \cup U^{(n)}_{i_1}(\{\omega_3\}) = \{\omega_3\} \supsetneq \emptyset = U^{(n)}_{i_1}(\Omega)$. Second, consider the "$\supseteq$" part. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and $\mathcal{J} = \{\emptyset, \{\omega_1, \omega_2\}, \Omega\}$. We have: $U^{(2)}(\{\omega_1, \omega_2\}) = (\neg K)(\{\omega_1, \omega_2\}) \cap (\neg K)^2(\{\omega_1, \omega_2\}) = \{\omega_3\}$; $U^{(2)}(\{\omega_1\}) = (\neg K)(\{\omega_1\}) \cap (\neg K)^2(\{\omega_1\}) = \emptyset$; and $U^{(2)}(\{\omega_2\}) = (\neg K)(\{\omega_2\}) \cap (\neg K)^2(\{\omega_2\}) = \emptyset$. Hence, $U^{(2)}(\{\omega_1, \omega_2\}) \supsetneq U^{(2)}(\{\omega_1\}) \cup U^{(2)}(\{\omega_2\})$. Next, let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $\mathcal{J} = \{\emptyset, \{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_1, \omega_2, \omega_3\}, \Omega\}$. See Table 2 below. Then, it can be seen that $U^{(\infty)}(\{\omega_2, \omega_3, \omega_4\}) = \{\omega_4\} \not\subseteq \emptyset = U^{(\infty)}(\{\omega_2\}) \cup U^{(\infty)}(\{\omega_3\}) \cup U^{(\infty)}(\{\omega_4\})$.

3. Consider $U^{(n)}(\bigcap_{\lambda\in\Lambda} E_\lambda) = \bigcap_{\lambda\in\Lambda} U^{(n)}(E_\lambda)$. We have $U^{(n)}_{i_1}(\{\omega_1, \omega_3\} \cap \{\omega_2, \omega_3\}) = \emptyset \neq \{\omega_3\} = U^{(n)}_{i_1}(\{\omega_1, \omega_3\}) \cap U^{(n)}_{i_1}(\{\omega_2, \omega_3\})$. Also, $U^{(n)}_{i_1}(\Omega \cap \{\omega_1\}) = \{\omega_3\} \supsetneq \emptyset = U^{(n)}_{i_1}(\Omega) \cap U^{(n)}_{i_1}(\{\omega_1\})$.

E	K	$(\neg K)$	$(\neg K)^2$	$(\neg K)^3$	∂	$U^{(2)}$	$U^{(n)}$
$\emptyset$	$\emptyset$	Ω	$\emptyset$	Ω	$\emptyset$	$\emptyset$	$\emptyset$
$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_1,\omega_4\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_4\}$	$\{\omega_4\}$	$\{\omega_4\}$
$\{\omega_2\}$	$\emptyset$	Ω	$\emptyset$	Ω	$\{\omega_2,\omega_3,\omega_4\}$	$\emptyset$	$\emptyset$
$\{\omega_3\}$	$\emptyset$	Ω	$\emptyset$	Ω	$\{\omega_2,\omega_3,\omega_4\}$	$\emptyset$	$\emptyset$
$\{\omega_4\}$	$\emptyset$	Ω	$\emptyset$	Ω	$\{\omega_4\}$	$\emptyset$	$\emptyset$
$\{\omega_1,\omega_2\}$	$\{\omega_1\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_1,\omega_4\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_4\}$	$\{\omega_4\}$
$\{\omega_1,\omega_3\}$	$\{\omega_1\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_1,\omega_4\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_4\}$	$\{\omega_4\}$
$\{\omega_1,\omega_4\}$	$\{\omega_1\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_1,\omega_4\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_4\}$	$\{\omega_4\}$	$\{\omega_4\}$
$\{\omega_2,\omega_3\}$	$\{\omega_2,\omega_3\}$	$\{\omega_1,\omega_4\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_1,\omega_4\}$	$\{\omega_4\}$	$\{\omega_4\}$	$\{\omega_4\}$
$\{\omega_2,\omega_4\}$	$\emptyset$	Ω	$\emptyset$	Ω	$\{\omega_2,\omega_3,\omega_4\}$	$\emptyset$	$\emptyset$
$\{\omega_3,\omega_4\}$	$\emptyset$	Ω	$\emptyset$	Ω	$\{\omega_2,\omega_3,\omega_4\}$	$\emptyset$	$\emptyset$
$\{\omega_1,\omega_2,\omega_3\}$	$\{\omega_1,\omega_2,\omega_3\}$	$\{\omega_4\}$	Ω	$\emptyset$	$\{\omega_4\}$	$\{\omega_4\}$	$\emptyset$
$\{\omega_1,\omega_2,\omega_4\}$	$\{\omega_1\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_1,\omega_4\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_4\}$	$\{\omega_4\}$
$\{\omega_1,\omega_3,\omega_4\}$	$\{\omega_1\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_1,\omega_4\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_4\}$	$\{\omega_4\}$
$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_2,\omega_3\}$	$\{\omega_1,\omega_4\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_1,\omega_4\}$	$\{\omega_4\}$	$\{\omega_4\}$	$\{\omega_4\}$
Ω	Ω	$\emptyset$	Ω	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Table 2: Violation of $U^{(n)}(\bigcup_{\lambda\in\Lambda} E_\lambda) \subseteq \bigcup_{\lambda\in\Lambda} U^{(n)}(E_\lambda)$

Proof of Proposition 4. First, we show the equivalence between (a) and (b). If (a) holds, then $K(\Omega) \setminus E = K(\Omega) \setminus K(E) \in \mathcal{J}$ for any $E \in \mathcal{J}$, i.e., (b) holds. Conversely, if (b) holds then $K(E) \in \mathcal{J}$ implies (a).

Second, we show the equivalence between (a) and $U^{(2)}(\cdot) = (\neg K)(\Omega)$. Suppose (a). Since $K(\neg K)(E) = K(\Omega \cap (KE)^c) \supseteq K(K(\Omega) \setminus K(E))$, we have $(\neg K)^2(E) \subseteq (\neg K)(K(\Omega) \setminus K(E)) = (K(\Omega) \setminus K(E))^c$. Then, we get

$$\begin{aligned} U^{(2)}(E) = (\neg K)(E) \cap (\neg K)^2(E) &\subseteq ((\neg K)(\Omega) \cup (K(\Omega) \setminus K(E))) \cap (K(\Omega) \setminus K(E))^c \\ &= (\neg K)(\Omega). \end{aligned}$$

Since $(\neg K)(\Omega) \subseteq U^{(2)}(E)$, we obtain $U^{(2)}(E) = (\neg K)(\Omega)$. Conversely, if $U^{(2)}(E) = (\neg K)(\Omega)$, then we have $K(\Omega) = K(E) \cup K(\neg K)(E)$. Then, (a) follows because

$$K(\Omega) \cap (KE)^c = (K(E) \cup K(\neg K)(E)) \cap (\neg K)(E) = K(\neg K)(E) \in \mathcal{J}.$$

Third, if $U^{(2)}(E) = (\neg K)(\Omega)$, then $(\neg K)(\Omega) = (\neg K)(E) \cap (\neg K)^2(E)$. Since $(\neg K)(\Omega) \subseteq (\neg K)^3(E)$, we obtain (c). Clearly, (c) implies (d). Now, (d) implies that $U^{(2)}A^{(2)} = U^{(\infty)}A^{(2)}$. It follows from Proposition 3 that $U^{(2)}A^{(2)} = (\neg A^{(2)})A^{(2)} = (\neg A^{(2)}) = U^{(2)}$. On the other hand, it also follows from Proposition 3 that $U^{(\infty)}A^{(2)}(E) = U^{(2)}(\neg K)A^{(2)}(E) = U^{(2)}U^{(2)}(E) = (\neg K)(\Omega)$. Thus, we obtain $U^{(2)}(E) = (\neg K)(\Omega)$.

Fourth, if $U^{(2)}(\cdot) = (\neg K)(\Omega)$, then Symmetry (i.e., (e)) is trivially satisfied. Conversely, if Symmetry holds, then $U^{(2)}(E) = U^{(2)}(E) \cap U^{(2)}(E^c) \subseteq U^{(\infty)}(E) \subseteq U^{(2)}(E)$, where the first set-inclusion follows from Proposition A.1. We get (d).

Fifth, if $U^{(n)}(\cdot) = (\neg K)(\Omega)$, then AU Introspection (i.e., (f)) is clearly satisfied. If AU Introspection (i.e., (f)) holds, then together with Reverse AU Introspection and Weak Necessitation in Proposition 3, we obtain $U^{(n)}(\cdot) = U^{(n)}U^{(n)}(\cdot) = (\neg K)(\Omega)$.

Alternatively, we show that Symmetry (i.e., (e)) and AU Introspection (i.e., (f)) are equivalent. If (e) holds, then we have $U^{(2)}(E) = (\neg A^{(2)})(E) = (\neg A^{(2)})A^{(2)}(E) = U^{(2)}A^{(2)}(E) = U^{(2)}U^{(2)}(E)$, where (e) is used for the last equality. This implies (f), which, in turn, implies $U^{(2)}(\cdot) = (\neg K)(\Omega)$. Conversely, (f) and Proposition 3 imply that $U^{(2)}(E) = U^{(2)}U^{(2)}(E) = (\neg K)(\Omega)$. Then, (e) trivially holds.

Sixth, if $U^{(n)}(\cdot) = (\neg K)(\Omega)$, then $U^{(n)}$ (resp. $A^{(n)}$) is obviously monotonic. Conversely, since $U^{(n)}(\Omega) = (\neg K)(\Omega) = U^{(n)}(\emptyset)$, Monotonicity of $U^{(n)}$ implies that $U^{(n)}(\cdot) = (\neg K)(\Omega)$. Likewise, since $A^{(n)}(\Omega) = A^{(n)}(\emptyset) = K(\Omega)$, Monotonicity of $A^{(n)}$ implies that $A^{(n)}(\cdot) = K(\Omega)$, i.e., $U^{(n)}(\cdot) = (\neg K)(\Omega)$.

Ninth, $U^{(\infty)}(\cdot) = (\neg K)(\Omega)$ clearly implies AA Self-Reflection (i.e., (i)). Conversely, observe that

$$\begin{aligned} A^{(\infty)}A^{(\infty)}(E) &= K(\neg K)A^{(\infty)}(E) \cup K(\neg K)^2A^{(\infty)}(E) \\ &= KU^{(\infty)}(E) \cup K(\neg K)U^{(\infty)}(E) = K(\Omega). \end{aligned}$$

Then, (i) implies that $A^{(\infty)}(E) = A^{(\infty)}A^{(\infty)}(E) = K(\Omega)$, i.e., $U^{(\infty)}(E) = (\neg K)(\Omega)$. □

A.4 Section 4.1

Proof of Proposition 5. 1. Suppose that $\mathcal{S}^{(n)}$ satisfies Finite Conjunction. First, it follows from Lemma A.2 and Positive Introspection that $K(E \cup F) \subseteq K(KE \cup$

LF) for any $E, F \in \mathcal{D}$. Second, since $\mathcal{D}$ is assumed to be finite, we re-label it by $\mathcal{D} = \{E_1, \ldots, E_m\}$, so that $\overline{U}^{(n)} = \bigcup_{r=1}^{m} U^{(n)}(E_r)$. Putting $E = U^{(n)}(E_m)$ and $F = \bigcup_{r=1}^{m-1} U^{(n)}(E_r)$ in the previous statement yields

$$K(\bigcup_{r=1}^{m} U^{(n)}(E_r)) \subseteq K(KU^{(n)}(E_m) \cup L(\bigcup_{r=1}^{m-1} U^{(n)}(E_r)))$$
$$= K(KU^{(n)}(E_m) \cup \bigcup_{r=1}^{m-1} U^{(n)}(E_r)) = K(\bigcup_{r=1}^{m-1} U^{(n)}(E_r)),$$

where observe that $\bigcup_{r=1}^{m-1} U^{(n)}(E_r) = L(\bigcup_{r=1}^{m-1} U^{(n)}(E_r))$ follows from Finite Conjunction. It follows, by induction, that $K(\overline{U}^{(n)}) \subseteq K(U^{(n)}(E_1)) = \emptyset$. Thus, we have $K(\neg K)(\overline{U}^{(n)}) = K(\Omega)$. By Monotonicity of K, we have $A^{(n)}(\overline{U}^{(n)}) = K(\Omega)$.

Next, we provide a counterexample when $\mathcal{D}$ is not finite. Let $\Omega = \mathbb{R}$ and $\mathcal{D} = \mathcal{P}(\Omega)$. Suppose that an agent's self-evident collection is given by the usual Euclidean topology $\mathcal{E}_{\mathbb{R}}$. Since it is closed under arbitrary union and finite intersection, her knowledge satisfies Finite Conjunction. Her knowledge also satisfies Necessitation because $\Omega \in \mathcal{E}_{\mathbb{R}}$. For any $\omega \in \mathbb{R}$, let $E_\omega = (\omega, +\infty)$. Then, we have $U^{(2)}(E_\omega) = \partial K E_\omega = \{\omega\}$ and $U^{(\infty)}(E_\omega) = \partial LKE_\omega = \{\omega\}$. Thus, we have $K(\bigcup_{\omega \in \Omega} U^{(n)}(E_\omega)) = K(\Omega) = \Omega$. This implies that $K(\overline{U}^{(n)}) = \Omega$.

2. If $K(\overline{U}^{(n)}) = \emptyset$, then we have $K(\Omega) \supseteq A^{(n)}(\overline{U}^{(n)}) \supseteq K(\neg K)(\overline{U}^{(n)}) = K(\Omega)$. Thus, $A^{(n)}(\overline{U}^{(n)}) = K(\Omega) \neq \emptyset$. If $K(\overline{U}^{(n)}) \neq \emptyset$, then $A^{(n)}(\overline{U}^{(n)}) \supseteq K(\overline{U}^{(n)}) \neq \emptyset$. Next, it is clear by construction that $U^{(n)}(\overline{U}^{(n)}) \subseteq \bigcup_{E \in \mathcal{D}} U^{(n)}(E) = \overline{U}^{(n)}$.

3. If $\overline{U}^{(n)} = \emptyset$ then Necessitation implies that we have $\overline{U}^{(n)} = \emptyset = L(\emptyset) = L(\overline{U}^{(n)})$. If $\overline{U}^{(n)} \neq \emptyset$, then (Non-empty) Conjunction (of K) and A-Introspection (see Proposition 3) imply that $L(\overline{U}^{(n)}) = L(\bigcup_{E \in \mathcal{D}} U^{(n)}(E)) = \bigcup_{E \in \mathcal{D}} LU^{(n)}(E) = \bigcup_{E \in \mathcal{D}} U^{(n)}(E) = \overline{U}^{(n)}$.

□

A.5 Section 4.3

Proof of Proposition 6. 1. Consider (1a) and (1b). Since $\partial_j(\cdot) \subseteq \partial_i(\cdot)$, substitute $K_i(E)$ and $L_i K_i(E)$. Consider (1c). First, we have $\partial_j U_i^{(n)}(E) = (\neg K_j) U_i^{(n)}(E) \cap (\neg K_j) A_i^{(n)}(E) \subseteq (\neg K_j) A_i^{(n)}(E) = U_i^{(n)}(E)$. Second, we have $\partial_i U_j^{(n)}(E) = (\neg K_i) U_j^{(n)}(E) \cap (\neg K_i) A_j^{(n)}(E) = (\neg K_i) A_j^{(n)}(E) \supseteq (\neg K_j) A_j(E) = U_j^{(n)}(E)$.

2. Substituting $K_j(E)$ and $K_j(\neg K_j)(E)$ into $\partial_j(\cdot) \subseteq \partial_i(\cdot)$ yields the desired results.

3. First, we have $A_i^{(n)}(E) = K_i A_i^{(n)}(E) \subseteq K_j A_i^{(n)}(E) \subseteq A_i^{(n)}(E)$. Second, noting that $K_j K_i(E) = K_i(E)$, we have $A_i^{(n)}(E) = A_i^{(n)} K_i(E) = A_i^{(n)} K_j K_i(E)$.

4. We have $A_i^{(n)}(E) \subseteq K_i A_i^{(n)}(E) \subseteq K_j A_i^{(n)}(E) \subseteq A_j^{(n)} A_i^{(n)}(E)$. Next, by (1c) and Proposition 1, $U_j^{(n)} U_i^{(n)}(E) \subseteq \partial_j U_i^{(n)}(E) \subseteq U_i^{(n)}(E)$.

□

Proof of Proposition 7. Since $U_i^{(\infty)} = U_i^{(3)}$, we can let $n \in \{2, 3\}$.

1. If $\omega' \in U_i^{(n)}(E) = \bigcap_{r=1}^{n} (\neg K_i)^r$, then $K_i(\neg K_i)^{r-1}(E) \notin \mathrm{IK}_i(\omega')$ for all $r \leq n$. Since $\mathrm{IK}_i(\omega) \subseteq \mathrm{IK}_i(\omega')$, we have $K_i(\neg K_i)^{r-1}(E) \notin \mathrm{IK}_i(\omega)$ for all $r \leq n$, that is, $\omega \in U_i^{(n)}(E) = \bigcap_{r=1}^{n} (\neg K_i)^r$.

2. Suppose that $\omega \in U_j^{(n)}(E)$. If $\omega \in K_i U_j^{(n)}(E)$, then we obtain $\omega \in K_i U_j^{(n)}(E) \subseteq K_j U_j^{(n)}(E) = \emptyset$, a contradiction. Hence, $\omega \notin K_i U_j^{(n)}(E)$. Next, suppose that $\omega \in A_i^{(n)}(E)$. Then, we have $\omega \in A_i^{(n)}(E) = K_i A_i^{(n)}(E) \subseteq K_j A_i^{(n)}(E)$.

□

B Extensions to Subjective State Spaces

Let Ω be a state space. Let $\mathcal{D}_i$ and be a complete algebra on *a subset* Ω_i of Ω. Note that $\emptyset = \bigcup \emptyset \in \mathcal{D}$ and $\Omega_i = \bigcap \emptyset \in \mathcal{D}_i$. Now, we define player i's knowledge and unawareness operators on $\mathcal{D}_i$. An information structure is a tuple $\mathcal{S} = \langle \Omega, (\mathcal{D}_i)_{i \in I}, (K_i, U_i)_{i \in I} \rangle$, where $K_i : \mathcal{D}_i \to \mathcal{D}_i$ and $U_i : \mathcal{D}_i \to \mathcal{D}_i$.

Let $\mathcal{D}$ be a complete algebra on Ω such that $\mathcal{D}_i \subseteq \mathcal{D}$ for all $i \in I$. We re-define players' knowledge and unawareness operators on this common domain $\mathcal{D}$ from $\mathcal{D}_i$. Specifically, we take the following two approaches. One approach is the logical approach in which we extend players' knowledge and unawareness operators by rendering players the logical inference ability to judge whether they know events outside of their domains. The other approach is the naive approach, where each player is assumed not to know any event which is outside of her original domain $\mathcal{D}_i$.

We make the following three technical remarks. The first is with respect to the difference between Ω and Ω_i. Although we define Ω as the entire state space, objects of knowledge and unawareness, from the standpoint of each player i, are defined only on Ω_i. Thus, each player i would consider any state $\omega \in \Omega \setminus \Omega_i$ "impossible" whenever $\Omega \setminus \Omega_i \neq \emptyset$. In other words, each Ω_i is player i's subjective state space.

Second, from each player's perspective, the complementation is always taken with respect to Ω_i as a universal set. Hence, if we take the complement of a set with respect to Ω_i, we often append the subscript i by denoting $\neg_i E := \{\omega \in \Omega_i \mid \omega \notin E\}$ for $E \in \mathcal{P}(\Omega_i)$, in contrast to $\neg E := \{\omega \in \Omega \mid \omega \notin E\}$ for $E \in \mathcal{P}(\Omega)$. We also denote $E^{c_i} := \neg_i E$ and $E^c := \neg E$.

Third, the distinction between Ω and Ω_i allows us to examine the implications of Conjunction and Negative Introspection independently of Necessitation. Namely, either Conjunction or Negative Introspection implies $K_i(\Omega_i) = \Omega_i$, which does not always imply Necessitation in the form of $K_i(\Omega) = \Omega$ when $\Omega_i \subsetneq \Omega$. Note that Negative Introspection implies that $(\neg_i K_i)(E) \subseteq K_i(\neg_i K_i)(E)$ (or $\Omega_i \setminus K_i E \subseteq K_i(\Omega_i \setminus K_i E)$) in Ω_i. Conjunction is also taken with respect to Ω_i. Thus, we further introduce the following definition: K_i satisfies Objective Necessitation if $K_i(\Omega) = \Omega$. Note that if Objective Necessitation is assumed, then it is implicitly assumed that $\Omega = \Omega_i \in \mathcal{D}_i$.

B.1 Retaining Logical Ability

Recall that each player's knowledge is represented by her self-evident collection $\mathcal{J}_i(\subseteq \mathcal{D}_i)$ in the sense that $K_i(E) = \{\omega \in \Omega_i \mid \text{there is } F \in \mathcal{J}_i \text{ such that } \omega \in F \subseteq E\}$. That is, player i knows an event $E \in \mathcal{D}_i$ at a state $\omega \in \Omega_i$ iff she infers E from a self-evident event $F \in \mathcal{J}_i(\subseteq \mathcal{D}_i)$ which is true at ω. Hence, for any event $E' \in \mathcal{D}$, we define that she knows E' at state $\omega' \in \Omega$ if she infers E' from a self-evident event $F \in \mathcal{J}_i(\subseteq \mathcal{D})$ which is true at ω'.

Formally, we define $\tilde{K}_i : \mathcal{D} \to \mathcal{D}$ by $K_i(E) = \{\omega \in \Omega \mid \text{there is } F \in \mathcal{J}_i(\subseteq \mathcal{D}) \text{ such that } \omega \in F \subseteq E\}$. Then, we have $\tilde{K}_i(E) = K_i(E \cap \Omega_i)$ for any $E \in \mathcal{D}$ such that $\Omega_i \cap E \in \mathcal{D}_i$. For a given knowledge operator K_i, the new operator inherits its properties except for Necessitation (Objective Necessitation has to be explicitly assumed, if necessary).

Our previous arguments can then be applied, and hence this logical extension justifies the common domain assumption within our framework. Note that for single player's analysis (in Sections 3 and 4.2), the information structure $\mathcal{S} = \langle(\Omega, \mathcal{D}), (K, U^{(n)})\rangle$ should rather be read as $\mathcal{S} = \langle \Omega_i, \mathcal{D}_i, (K_i, U_i^{(n)})\rangle$.

As a specific example, consider players i_3 and i_4 in Example 1. Suppose that $\mathcal{D} = \mathcal{D}_{i_3} = \mathcal{P}(\Omega)$ and $\mathcal{D}_{i_4} = \{\omega_1\}$. Let $K'_{i_3} : \mathcal{D}_{i_3} \to \mathcal{D}_{i_3}$ be defined by $K'_{i_3} = K_{i_3}$. Let $K'_{i_4} : \mathcal{D}_{i_4} \to \mathcal{D}_{i_4}$ be defined by $K'_{i_4} = (\emptyset) = \emptyset$ and $K'_{i_4}(\{\omega_1\}) = \{\omega_1\}$. Then, it is easily seen that $\tilde{K'}_{i_4} : \mathcal{D} \to \mathcal{D}$ is given by K_{i_4} (the original knowledge operator specified in Example 1). If we assume Objective Necessitation then $\tilde{K'}_{i_4}$ is given by K_{i_3}.

B.2 Introducing Non-monotonicity

The second approach is to assume that each player simply does not know $E \in \mathcal{D} \setminus \mathcal{D}_i$. This breaks down players' logical ability in terms of Monotonicity. Formally, we define $\overline{K}_i : \mathcal{D} \to \mathcal{D}$ by

$$\overline{K}_i(E) := \begin{cases} K_i(E) & \text{if } E \in \mathcal{D}_i \\ \emptyset & \text{if } E \in \mathcal{D} \setminus \mathcal{D}_i \end{cases}.$$

Next, we define $\overline{U}_i^{(n)} : \mathcal{D} \to \mathcal{D}$ by $\overline{U}_i^{(n)}(E) := \bigcap_{r=1}^{n} (\neg \overline{K}_i)^r(E)$. Thus, we have:

$$\overline{U}_i^{(n)}(E) = \begin{cases} U_i^{(n)}(E) & \text{if } E \in \mathcal{D}_i \\ \Omega & \text{if } E \in \mathcal{D} \setminus \mathcal{D}_i \end{cases}.$$