The Existence of Universal Knowledge Spaces*

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Abstract

We provide a formal framework capable of capturing players' interactive knowledge in a strategic context with a number of desirable features. First, we can specify players' logical and introspective abilities as well as the language that they can use in their reasoning. Second, our framework admits tractable representations of players' knowledge and common knowledge and nests previous interactive knowledge models. The main result of the paper is that the framework admits a canonical representation of players' knowledge. The canonical model is a "largest" model of knowledge to which any particular knowledge space can be mapped in a unique knowledge-preserving way.

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Contents

1 Introduction

Consider a group of players who reason interactively about unknown parameters, such as the payoffs and strategies in a game. The value of such exogenous parameters is called a state of nature. Each player must reason about states of nature and about each other's knowledge about states of nature, and so on. In this paper, we construct a first formal framework general enough to represent any conceivable form of such interactive knowledge. An arbitrary structure will capture some possible aspects of players' interactive reasoning but will generally exclude others. We construct a sufficiently rich knowledge space that includes all possible forms of reasoning about knowledge. The construction enables analysis of formal questions regarding knowledge without ad hoc restrictions on the nature of reasoning. Then claims regarding knowledge can be logically disassociated from extraneous structure that is not immediately related to interactive knowledge.

A model of knowledge consists of the following three ingredients. The first ingredient is a set Ω. Each element $\omega \in \Omega$ is a list of possible specifications of the realization of the exogenous parameters (i.e., the prevailing state of nature $s \in S$, where S is the set of unknown parameter values) and players' interactive knowledge regarding states of nature S (i.e., their knowledge about states of nature S, their knowledge about their knowledge about S, and so on). Call each specification ω a state of the world.

The second ingredient is the set of statements about which the players can reason. These statements, which we call events of the world, are specified as subsets of states of the world Ω. Thus a model of knowledge requires a description of the language available to the players, modeled as a collection of subsets of Ω which we call the domain.

The third ingredient is players' knowledge operators defined on the domain. For each event of the world E, player i's knowledge operator assigns the set of states at which she knows E, i.e., the event that i knows E. By iterative application of this operator, we can unpack higher orders of interactive reasoning.

In sum, a model of knowledge (a knowledge space) consists of the description of states of the world, the description of events of the world, and players' knowledge operators.

In applications, typically a specific model of knowledge is assumed a priori. This leaves open the possibility that some relevant aspects of reasoning are excluded. To address this, we construct a canonical representation of players' knowledge. This space is universal in the sense that any other knowledge space is embedded in it in a unique manner that maintains all the structure of that smaller space. That is, any form of reasoning in the smaller space can be retrieved in the universal space in a unique way. At the same time, we prove that the space is complete in the sense of including all possible forms of reasoning. It reveals what form of reasoning is indeed lost in the specific smaller space.

The main result of the paper (Theorem 1 in Section 3) is to demonstrate the existence of a universal knowledge space. The existence of a universal knowledge space ensures that players' interactive knowledge in a strategic situation can be modeled by the knowledge space approach without neglecting any form of reasoning.

We also demonstrate the applicability of our framework. First, we establish the existence of a universal knowledge space under a variety of assumptions on players' logical and introspective abilities. Our result is theoretically interesting in that the existence of a universal knowledge space is unrelated to assumptions on players' knowledge. At the same time, it is substantively interesting because we establish the canonical representation of knowledge even when players are less than "perfectly rational" in terms of their logical and introspective abilities.

Second, we show (in Section 4) that our framework subsumes alternative representations of knowledge and common knowledge.[1] We demonstrate that our framework nests previous frameworks. Thus, our approach explicitly clarifies what implicit assumptions on players' knowledge are imposed in a specific previous framework. We can also reproduce existent theorems regarding knowledge and common knowledge obtained in a specific previous framework.

We identify the conditions on players' knowledge operators under which their knowledge is represented as information sets on the underlying states of the world, where each player's information set associated with a state of the world represents the set of states of the world she considers possible at that state. That is, our framework nests the most standard model of knowledge, known as a partition model of knowledge (Aumann [3]).

In relation to the first point, we can also accommodate other previous models of knowledge. For example, the players may not fully be introspective as to what they do not know. Our framework nests non-partitional models (also called possibility correspondence models) of knowledge which attempt to relax this introspective property called Negative Introspection.[2] There, each player's knowledge at each state of the world is represented as a general information set, which may not necessarily constitute a partition of the states of the world. The study of non-partitional models ranges from implications of common knowledge (e.g., Agreement theorems (Aumann [3])) to solution concepts in game theory.[3]

[1]Pioneering attempts to model common knowledge include, but are not limited to, Aumann [3], Friedell [29], Lewis [48], and McCarthy, Sato, Hayashi, and Igarashi [53]. We show that various previous formalizations of common knowledge can be nested in our framework. Hence, we can analyze, within our framework, implications of common knowledge on strategic situations.

[2]Non-partitional models are motivated in part by notions of unawareness. The pioneering studies of unawareness include Fagin and Halpern [27], Modica and Rustichini [62, 63], and Pires [70]. See also Schipper [79] for a recent overview of the field. While Modica and Rustichini [62] and Dekel, Lipman, and Rustichini (hereafter, DLR) [21] demonstrate that standard state space models (including non-partitional models) have limitations in representing unawareness, in a companion paper (Fukuda [30]), we study how standard state space models can (and cannot) represent notions of unawareness within our framework.

[3]See, for example, Bacharach [6], Binmore and Brandenburger [11], Brandenburger, Dekel, and Geanakoplos (hereafter, BDG) [17], Dekel and Gul [20], Geanakoplos [31], Morris [64], Rubinstein

Moreover, our framework also nests other forms of possibility correspondence models. One such example is a situation where players' knowledge may not necessarily be true.[4] Thus we can deal with qualitative belief as well as knowledge. Indeed, we can endow each player with both (qualitative) belief and knowledge.[5] Our general model clarifies the assumptions regarding logical sophistication that are formally implicit in possibility correspondence models. We can further relax players' logical abilities in that they can fail to know a logical consequence of their knowledge.

Third, while players' interactive knowledge is an important consideration in strategic contexts, epistemic analyses of strategic situations call for both knowledge and probabilistic belief.[6] Our framework admits a probability (precisely, a measurable) space as a domain of players' knowledge, and thus we can incorporate players' probabilistic assessments into the framework. Indeed, the consideration of the domain of a knowledge space has often been neglected. In standard models of knowledge in the previous literature, any subset of states of the world Ω is considered to be an object of knowledge. Under such specification, knowledge and probabilistic beliefs may be incompatible with each other when the knowledge of an event is not in the domain of the probability space. This incompatibility has been one of the issues that have hampered the epistemic analyses of players' knowledge and belief.

Fourth, we pose the following conceptual question: can we formalize the sense in which the players know (or commonly know) the structure of a model of knowledge itself?[7] We provide a formal test (in Section 5) according to which we (i.e., the outside analysts) can say that the players know (indeed, commonly know) the structure of a model of knowledge itself, provided that they are introspective as to their own knowledge and that assumptions on their knowledge are homogeneous. In this test, we regard each player's knowledge as a signal and ask whether each player knows the signal that generates each other's knowledge. We show that each player knows her

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and Wolinsky [72], Samet [75, 76], and Shin [80].

[4]In the literature, knowledge is qualitatively distinguished from belief in that a player can only know what is true while she can believe something false. Since we deal with a wide variety of assumptions on "knowledge" at the same time, by abusing the terminologies, we refer to such a belief as knowledge in a generic context. In a specific application, however, the distinction between knowledge and belief should be made.

[5]Such consideration would be needed if, for example, we analyze each player's knowledge about her own strategy and her belief about her opponents' strategies (e.g., Dekel and Gul [20]). The analysis of an extensive form game would also call for knowledge and belief if we analyze players' knowledge about their past observed moves and their belief about past unobserved moves and future moves (e.g., Battigali and Bonanno [9]). See also Dekel and Gul [20, Section 5].

[6]Once we incorporate both knowledge and probabilistic belief, a distinction between knowledge and probability-one belief would emerge in a continuous model. For example, an agent believes with probability one that a random number drawn from the interval $[0, 1]$ is irrational while she does not know it (Monderer and Samet [61]).

[7]This question has been puzzling a number of economists and game theorists. See such discussions by Aumann [3, 4, 5], Bacharach [6, 7], Binmore and Brandenburger [11], Brandenburger and Dekel [16], BDG [17], Dekel and Gul [20], Fagin, Geanakoplos, Halpern, and Vardi (hereafter, FGHV) [26], Gilboa [32], Pires [70], Roy and Pacuit [71], Tan and Werlang [82].

own signal when her knowledge is introspective. In the universal knowledge space, moreover, no relevant aspect of players' interactive knowledge is left unspecified. If assumptions on players' knowledge are homogeneous, each player has an identical knowledge operator. We show that the players (in a formal sense) know (and commonly know) each other's signal that generates their knowledge.

In conclusion, we provide a framework capable of capturing players' interactive knowledge in the following ways: (i) the framework contains a universal knowledge space; (ii) the framework admits a wide variety of assumptions on players' logical and introspective abilities; and (iii) the framework allows us to specify the language that the players can use through selecting feasible domains. Moreover, the framework is amenable to tractable representations of players' knowledge and common knowledge in the sense that it nests and generalizes previous models of interactive knowledge.

The paper is organized as follows. The rest of this section reviews the previous literature and provides a technical overview of the main result. Section 2 defines a knowledge space, properties of knowledge, and a universal knowledge space. Section 3 demonstrates that our framework admits a universal knowledge space (Theorem 1).

Section 4 provides tractable representations of players' knowledge and common knowledge. In Section 5, we formally ask the sense in which the players know the structure of a model of knowledge.

Section 6 carries out an alternative construction of a universal knowledge space (Theorem 3) by formalizing a notion of hierarchies of knowledge. Section 7 provides applications to richer settings involving dynamics and other epistemic notions. Throughout the paper, all the proofs are relegated to Appendix A.

1.1 Literature Review

The original economic model of knowledge is Aumann's [3] partitional model. While the partitional approach has been prevalent, this approach poses the following three concerns.

First, a standard partitional knowledge space has, as its domain, the power set of Ω. This class of standard partitional knowledge spaces cannot be closed in the following sense: there is no standard partitional knowledge space that includes all standard partitional knowledge spaces. In the next subsection, we will discuss these negative results and how we circumvent them in more detail.

The second issue is the sophisticated logical and introspective abilities that the standard partitional models render to players. A player whose knowledge is dictated by a partition is logically omniscient. Moreover, she is fully introspective as to what she knows and what she does not know.

Third, partitions and probabilistic beliefs may be incompatible with each other. While players' probabilistic beliefs are defined on measurable events, the domain of knowledge is all subsets. Thus, players' knowledge may not be an object of their beliefs.

We can identify conditions on knowledge operators under which each player's knowledge is represented as a partition (more generally, a possibility correspondence). Thus, our framework nests partitional and non-partitional knowledge spaces in a way such that there is a universal partitional (non-partitional) knowledge space defined on a general domain.

How does the existence of a universal knowledge space relate to the various strands of the universal type space literature? Harsanyi [35] proposed the notion of type. Each player's type summarizes her probabilistic belief about exogenous parameter values and players' types (usually, types of her opponents) themselves. An arbitrary type space induces hierarchies of probabilistic beliefs. The question arises as to whether there exists a canonical (universal) type space which contains any other type space.

In their pioneering work, Armbruster and Böge [2], Böge and Eisele [12], and Mertens and Zamir [59] establish a universal belief/type space consisting of coherent hierarchies of beliefs with certain topological assumptions on unlderlying states of nature.[8]

Heifetz and Samet (hereafter, HS) [39], whose approach we follow, establish a universal type space without topological assumptions on states of nature. Their logical approach is extended to establishing a universal finitely additive belief space and a universal knowledge-belief space by Meier [55, 56].[9]

A special kind of a type structure is a possibility structure (Brandenburger [14] and Brandenburger and Keisler [18]) used in epistemic analyses of games. Possibility structures model players' probabilistic and non-probabilistic interactive beliefs. Mariotti, Meier, and Piccione [51] show the existence of a universal possibility structure by imposing topological restrictions on players' non-probabilistic beliefs. As a related

[8]Their results have been extended under weaker topological assumptions by Brandenburger and Dekel [16], Heifetz [36], Mertens, Sorin, and Zamir [58], and Pinter [69]. Pinter [69] dispenses with a topological assumption on states of nature in his construction of the universal type space consisting of coherent hierarchies. Instead, he imposes a suitable topology on each higher order space consisting of probability measures. Extensions to richer structures include conditional probability systems (Battigalli and Siniscalchi [10]) and ambiguous beliefs (Ahn [1]).

[9]The following two strands of literature also extend HS's [39] probabilistic universal type space. First, Moss and Viglizzo [65, 66] reformulate and generalize σ-additive type spaces as coalgebras for the endofunctor which we simply denote by F (see Moss and Viglizzo [65, Definition 2.1] for the precise definition of F). Then, a universal type space is reformulated as a final (terminal) coalgebra. They show that the set of descriptions of each point (type profile together with a state of nature) in all coalgebras, endowed with measurable and coalgebra structures, constitutes a final coalgebra. Furthermore, by invoking the "Lambek lemma" in category theory (Lambek [47]), there is an isomorphism between a final coalgebra T and its image of the endofunctor $F(T)$, which establsihes the "(belief-)completeness" of T (see Brandenburger [14] and Brandenburger and Keisler [18] for (belief-)completeness). Second, Meier [57] axiomatizes classes of belief/type spaces and shows that the space of all maximally consistent sets of formulas of his infinitary probability logic (i.e., the canonical space) is a universal space, which is isomorphic to the universal type space constructed by HS [39]. He also shows that, within the class of (product) type spaces, the canonical product type space is (belief-)complete.

model, Salonen [74] establishes a universal belief hierarchies where each agent's type induces a filter over basic propositions (which constitute a Boolean algebra) and the set of opponents' types.

Partly in order to make a formal connection between the knowledge space and type space approaches, we define in Section 5.1 a notion that we call a knowledge-type. Each knowledge-type is a mapping from events into the true-or-false statements about the knowledge of an event. Thus, each player's knowledge-type at each state describes her knowledge of events at that state. A knowledge-type mapping is then a mapping from a given set of states of the world to knowledge-types.[10] The given set of states of the world may not necessarily have a product structure. Specifying a knowledge-type mapping is equivalent to specifying a knowledge operator. In Section 6 (Theorem 3), by following HS's [39] hierarchical approach, we demonstrate a universal knowledge space in terms of hierarchies of knowledge-types. Each world in the resulting universal knowledge space consists of a state of nature and a profile of players' hierarchies of knowledge-types. We also characterize (in Theorem 4 in Section 6.2) our universal knowledge space in terms of coherent hierarchies of knowledge.

1.2 Technical Overview

HS [38] demonstrate that a universal standard partitional knowledge space generically does not exist, where recall that a standard partitional knowledge space allows any subset of underlying states of the world to be an object of knowledge. They show that, unlike σ-additive probabilistic beliefs, a non-trivial sequence of interactive knowledge can develop beyond any order (precisely, beyond any given ordinal number). The negative results are also obtained by Fagin [25], Fagin, Geanakoplos, Halpern, and Vardi (FGHV) [26], Fagin, Halpern, and Vardi (hereafter, FHV) [28], and HS [41], where they explicitly formalize notions of coherent hierarchies of knowledge. Moreover, Meier [54] shows, by invoking the result of HS [38] above, that there is no universal knowledge space even when knowledge is represented by a more general non-partitional structure. If there were a universal knowledge space in a class of such general knowledge spaces, then one could construct a universal partitional knowledge space from the given class, which is impossible.

How do our positive results reconcile with the negative results? What plays a crucial role in establishing a universal knowledge space is to specify a set algebra as objects of players' knowledge, i.e., a specification of the language that the players are allowed to use in their reasoning.

To see this point, let κ be an infinite cardinal number. We define a κ-complete algebra on underlying states of the world Ω to be a collection of subsets of Ω with the closure under complementation and under union and intersection of any sub-collection

[10] In Sections 5.2 and 5.3, we regard each player's knowledge-type mapping as a signal mapping that generates her knowledge in order to formally ask the (meta-)knowledge of the structure of a model.

with cardinality less than κ. Thus, if a certain set is an object of knowledge, then so is its complement. Also, if each of a collection of events is an object of knowledge, then so are its union and intersection, provided that the collection has cardinality less than κ. The power set of Ω is always κ-complete. For example, a κ-complete algebra subsumes an algebra of sets if κ is the least infinite cardinal number. Likewise, a κ-complete algebra subsumes a σ-algebra if κ is the least uncountable cardinal number. With this definition in mind, we call a knowledge space a κ-knowledge space if its domain is a κ-complete algebra.

Specifying the domain of each knowledge space by a κ-complete algebra amounts to determining the language available to the players in reasoning about their interactive knowledge regarding nature.[11] Namely, any κ-knowledge space can capture players' interactive knowledge of a form, player i knows that player j knows that ..., up to the level of κ. For example, any κ-knowledge space can capture any finite level of players' interactive knowledge when κ is the least infinite cardinal number. Likewise, any κ-knowledge space can capture any countable level of players' interactive knowledge when κ is the least uncountable cardinal number.

With this property in mind, we demonstrate (in Sections 3 and 6) that there is a universal κ-knowledge space by taking care of all the possible levels of interactive knowledge up to κ in a given class of κ-knowledge spaces. We do so for each class of knowledge spaces which respects given assumptions on players' knowledge.

Our construction has the following three implications. First, we circumvent the above mentioned non-existence results by explicitly specifying a domain of knowledge as a κ-complete algebra. On the one hand, the previously mentioned negative results imply that a sequence of interactive knowledge can generally develop beyond any depth of reasoning in a discontinuous way if any subset of states of the world is an object of knowledge. On the other hand, once we specify the language available to the players as a κ-complete algebra, any κ-knowledge space can generally take into consideration players' interactive knowledge up to the ordinality of κ.

Thus, we turn the previously mentioned negative results into the positive result in the following two ways. First, we enlarge a class of knowledge spaces by allowing the domain of a knowledge space to be a κ-complete algebra. Second, we find a universal κ-knowledge space by keeping track of all possible forms of reasoning up to the depth of κ attained in the given class of κ-knowledge spaces. Thus, unlike a universal (σ-additive) type space, our universal knowledge space usually has transfinite (precisely, κ) hierarchies of interactive knowledge incorporating all possible forms of interactive reasoning up to the depth of κ.

[11] It has been recognized as important in various strands of literature to explicitly endow the players with the language that they can (and cannot) use in their reasoning. Examples include: canonical representations of knowledge and belief, equivalence between syntactic and semantic models of knowledge and belief, and epistemic analyses of games. In relation to the first strand of literature, see, for instance, Aumann [5], Brandenburger [14], Brandenburger and Keisler [18], Fagin [25], FGHV [26], FHV [28], Gilboa [32], HS [39], Meier [55, 56, 57], Moss and Viglizzo [65, 66], Roy and Pacuit [71], Salonen [74], and Viglizzo [83].

Observe the following analogy with σ-additive beliefs. The domain of each type space is a σ-algebra because a σ-additive probability measure may not necessarily be defined on the power set. That is, the domain specification is implicitly incorporated in type spaces. Put differently, the domain specification plays the following role. The domain of any σ-additive type space (σ-algebra) is the language available to the players in reasoning up to any countable form of interactive beliefs. The domain of any $\aleph_1$-knowledge space (where $\aleph_1$-complete algebra is a σ-algebra because $\aleph_1$ is the least uncountable cardinal) is the language available to the players in reasoning up to any countable form of interactive knowledge. While we keep track of any form of interactive knowledge up to the ordinality of $\aleph_1$ in establishing a universal knowledge space, the continuity property of σ-additive beliefs (or more precisely, the continuity of the operation "Δ") guarantees that the least infinite ordinality of interactive beliefs suffices to establish a universal σ-additive type space (see, for example, FGHV [26] and HS [39] for this point). That is, the least infinite depth of interactive beliefs can determine any subsequent countable order of players' interactive beliefs. To elaborate on this point further, suppose that players' beliefs are finitely additive. Meier [55] shows, in a similar way to HS [38], that a universal finitely additive belief space does not exist if all subsets are required to be measurable.[12] On the other hand, Meier [55] also shows that a universal finitely additive belief space exists once players' beliefs are defined on a κ-complete algebra.

The second implication of our construction is that existence hinges on the specification of a domain rather than on the assumptions on players' knowledge. Third, we assert that there is a universal partitional (non-partitional) κ-knowledge space if we explicitly specify domains of such partitional (non-partitional) κ-knowledge spaces.

More specifically, our existence results are related to the following two previous positive results. First, Meier [56] demonstrates the existence of a universal knowledge-belief space when players' knowledge operators operate only on given measurable subsets of the space on which players' type distributions are defined. Our framework nests Meier's [56] (on the part of players' knowledge) as a special class of $\aleph_1$-knowledge spaces (where $\aleph_1$ is the least uncountable cardinal) when his assumptions on players' knowledge and states of nature are met. Note that the class of partitional $\aleph_1$-knowledge spaces is a proper subclass of his $\aleph_1$-knowledge spaces. We also study how we can obtain tractable representations of players' knowledge within these classes. For example, in both cases, each player's knowledge is represented as a σ-sub-algebra.

Second, Aumann [5] constructs what he calls a canonical knowledge system (of a finitary epistemic $S5$ logic), where each state of the world is a "complete and coherent" set of formulas describing finite levels of players' interactive knowledge. We formally show (in Theorem 2 in Section 3.2) how Aumann's [5] canonical knowledge system and our universal $\aleph_0$-knowledge space (where $\aleph_0$ is the least infinite cardinal) are related in terms of HS's [39] logical construction. Indeed, we do so for any combination of assumptions on players' knowledge and for any domain (i.e., for any κ).

[12] The related point is also made by FGHV [26, Example 4.5] in the context of belief hierarchies.

2 Knowledge Spaces

We denote by I a non-empty set of players. Let S be a set which we call the set of *states of nature*. An element of S is regarded as a specification of the exogenous parameters (e.g., strategies, payoff functions) that are relevant to the strategic interactions among the players. The set of states of nature S is endowed with a sub-collection $\mathcal{A}_S$ of $\mathcal{P}(S)$, where $\mathcal{P}(\cdot)$ is the power-set operation. Each element E of $\mathcal{A}_S$, called an event of nature, is an object about which players interactively reason.

2.1 Technical Preliminaries

Each event $E \in \mathcal{A}_S$ plays a role of a "proposition" regarding states of nature S. Thus, in order to specify a language the players are allowed to use in making inferences about states of nature S and their interactive knowledge, we endow $\mathcal{A}_S$ with a "logical" (precisely, a set-algebraic) structure. To that end, we introduce the following four technical definitions.

First, we denote by κ an infinite cardinal number. Second, for an infinite cardinal κ, we introduce the notion of a κ-complete (Boolean) algebra (of sets). For an underlying set Ω, a subset $\mathcal{D}$ of $\mathcal{P}(\Omega)$ is said to be a *κ-complete algebra* if $\mathcal{D}$ is closed under complementation and is closed under arbitrary union and intersection of any sub-collection with cardinality less than κ.[13] For example, an $\aleph_0$-complete algebra is an algebra of sets, because $\aleph_0$ is the least infinite cardinal. In the similar vein, an $\aleph_1$-complete algebra is a σ-algebra, because $\aleph_1$ is the least uncountable cardinal.

Note that we follow the conventions that $\emptyset = \bigcup \emptyset$ and that, with Ω being an underlying set, $\Omega = \bigcap \emptyset$. Hence, the requirement that any κ-complete algebra $\mathcal{D}$ on Ω contains $\emptyset$ and Ω is subsumed by the requirement that $\mathcal{D}$ is closed under the empty union and intersection.

Third, we say that a subset $\mathcal{D}$ of $\mathcal{P}(\Omega)$ is said to be a *complete algebra* if $\mathcal{D}$ is closed under complementation and is closed under arbitrary union and intersection. In order to conveniently refer to κ-complete and complete algebras, we also introduce a symbol ∞ so that, by abuse of notation, a complete algebra is also referred to as an ∞-complete algebra.

Fourth, for any given infinite cardinal κ or $\kappa = \infty$, we denote by $\mathcal{A}_\kappa(\cdot)$ the operation taking the smallest κ-complete algebra containing a given collection. That is, for any given collection $\mathcal{D}$ of subsets of an underlying set Ω, we define $\mathcal{A}_\kappa(\mathcal{D})$ by

$$\mathcal{A}_\kappa(\mathcal{D}) := \bigcap \{\mathcal{A} \in \mathcal{P}(\mathcal{P}(\Omega)) \mid \mathcal{A} \text{ is a } \kappa\text{-complete algebra with } \mathcal{D} \subseteq \mathcal{A}\}.$$

[13] First, such $\mathcal{D}$ is also called a κ-complete field (of sets) or a κ-field (e.g., Meier [55]). Second, we say that $\mathcal{D}$ is closed under (non-empty) κ-intersection if it is closed under intersection of a (non-empty) sub-collection with cardinality less than κ. Likewise, we say that $\mathcal{D}$ is closed under (non-empty) κ-union if it is closed under union of a (non-empty) sub-collection with cardinality less than κ.

With these four definitions in mind, for a given set of states of nature $(S, \mathcal{A}_S)$, we endow it with a κ-complete algebraic structure. Namely, we identify the given set of states of nature with $(S, \mathcal{A}_\kappa(\mathcal{A}_S))$.

This assumption means the following: (i) if E is an event of nature (i.e., an object of players' knowledge regarding states of nature S), then so is its complement E^c (we also denote the complement of E by $\neg E$); if E is an object of knowledge regarding S for each $E \in \mathcal{E}$ with $|\mathcal{E}| < \kappa$, then so are its union $\bigcup \mathcal{E} := \bigcup_{E \in \mathcal{E}} E$ and its intersection $\bigcap \mathcal{E} := \bigcap_{E \in \mathcal{E}} E$. Henceforth, we simply assume that $(S, \mathcal{A}_S)$ is a κ-complete algebra for a given κ, because our arguments hold by replacing $\mathcal{A}_S$ with $\mathcal{A}_\kappa(\mathcal{A}_S)$.

We have two further remarks regarding a κ-complete algebra $(S, \mathcal{A}_S)$. First, we usually assume that $\kappa > |I|$. This means that our "language" is fine enough to refer to statements regarding all the possible subsets of players. We will explicitly assume this assumption in Section 4.2, where we analyze the concepts of mutual and common knowledge among players.

Second, as mentioned in Meier [55, Remark 1], it is without loss of generality to restrict attention to κ-complete algebras for infinite regular cardinals κ or $\kappa = \infty$. If an infinite cardinal κ is not regular then any κ-complete algebra is indeed a κ^+-complete algebra, where κ^+ is the successor cardinal. Now, κ^+ is known to be regular (supposing the axiom of choice). Note also that $\aleph_0$ and $\aleph_1$ are indeed regular.

2.2 Knowledge Spaces in terms of Knowledge Operators

Now, we define a model of players' knowledge. We represent players' interactive knowledge regarding $(S, \mathcal{A}_S)$ on some "sample space" by knowledge operators.

Definition 1 (Knowledge Space)**.** *Let I be a non-empty set of players. Let $(S, \mathcal{A}_S)$ be a κ-complete algebra, where κ is an infinite (regular) cardinal or $\kappa = \infty$. A κ-knowledge space of I on $(S, \mathcal{A}_S)$ is a tuple $\overrightarrow{\Omega} := \langle (\Omega, \mathcal{D}), (K_i)_{i \in I}, \Theta \rangle$ with the following three properties.*

1. *$(\Omega, \mathcal{D})$ is a κ-complete algebra. Ω is the set of states of the world. Each element E of $\mathcal{D}$ is called an event (of the world).*
2. *$K_i : \mathcal{D} \to \mathcal{D}$ is player i's knowledge operator for each $i \in I$. For each $E \in \mathcal{D}$, the set $K_i(E)$ denotes the event that player i knows E (i.e., the set of states at which player i knows E). We say that a player $i \in I$ knows an event $E \in \mathcal{D}$ at a state $\omega \in \Omega$ if $\omega \in K_i(E)$.*
3. *$\Theta : \Omega \to S$ is a mapping such that $\Theta^{-1}(E) \in \mathcal{D}$ for any $E \in \mathcal{A}_S$. In other words, $\Theta : (\Omega, \mathcal{D}) \to (S, \mathcal{A}_S)$ is a κ-measurable mapping.*

We have three remarks regarding Definition 1. First, Condition (3) requires the inverse of Θ to be a well-defined mapping from $\mathcal{A}_S$ into $\mathcal{D}$. By this requirement, any set-algebraic ("logical") operations (taking intersections, unions, and complementation) in $\mathcal{A}_S$ are persevered (embedded) in the domain $\mathcal{D}$. In this regard, the mapping Θ can be regarded as a pair of mappings (Θ, Θ^{-1}) such (i) that Θ maps from Ω into S while Θ^{-1} maps $\mathcal{A}_S$ into $\mathcal{D}$ and (ii) that $\omega \in \Theta^{-1}(E)$ in $(\Omega, \mathcal{D})$ iff $\Theta(\omega) \in E$ in $(S, \mathcal{A}_S)$. Second, we do not impose any restriction on the cardinality of the sets S, $\mathcal{A}_S$, Ω, and $\mathcal{D}$. Third, we often call a κ-knowledge space $\overrightarrow{\Omega}$ of I on $(S, \mathcal{A}_S)$ to be a knowledge space $\overrightarrow{\Omega}$ by omitting κ, I, and $(S, \mathcal{A}_S)$.

While any subset of underlying states of the world is deemed to be an object of knowledge (i.e., $\mathcal{D} = \mathcal{P}(\Omega)$) in standard partitional models, our framework is more general. Especially, it intends to capture the following two cases.

First, it is often desirable to capture players' probabilistic beliefs in addition to their knowledge. To that end, we would like to have a model of knowledge whose domain is a certain set-algebra (such as a σ-algebra) in order to treat both knowledge and belief together. In contrast, suppose that players' probabilistic beliefs are represented on a σ-algebra of underlying states of the world while their knowledge is represented on the power set. Without additional assumptions on knowledge, however, the event that a player knows an E might not be measurable for a measurable event E.[14]

Second, in the literature giving logical foundations to state space models of knowledge, events are generated by some logical system, and thus the domain may only form a certain algebra of sets (depending on the given logical system).[15] In this regard, too, we would like to have a model of interactive knowledge amenable to such logical foundations.

Next, we define properties of knowledge to be analyzed.

Definition 2 (Propeties of Knowledge). *Let $\overrightarrow{\Omega} := \langle (\Omega, \mathcal{D}), (K_i)_{i \in I}, \Theta \rangle$ be a κ-knowledge space. Fix $i \in I$.*

1. *The following properties of K_i are referred to as logical properties.*

   (a) *No-Contradiction Axiom: $K_i(\emptyset) = \emptyset$.*

   (b) *Consistency: $K_i(E) \subseteq (\neg K_i)(E^c)$ for any $E \in \mathcal{D}$.*

   (c) *Monotonicity: $K_i(E) \subseteq K_i(F)$ for any $E, F \in \mathcal{D}$ with $E \subseteq F$.*

   (d) *Necessitation: $K_i(\Omega) = \Omega$.*

[14] For example, it is often assumed in standard partitional models that players' partitions are at most countable. See Maschler, Solan, and Zamir [52, Example 9.37] for an example where players' knowledge may not be measurable without such an assumption.

[15] Samet [78] is a model of knowledge whose primitive is a knowledge operator defined on an ($\aleph_0$-complete) algebra. Set-theoretical (semantic) models of knowledge where events are based on propositions include such papers as Aumann [5], Bacharach [6], Samet [75], and Shin [80]. In fact, it turns out later (in Section 3) that the domain of our universal knowledge space is generated by events corresponding to an "infinitary language" defined by nature and players' knowledge.

(e) *Non-empty λ-Conjunction:* $\bigcap_{E \in \mathcal{E}} K_i(E) \subseteq K_i(\bigcap \mathcal{E})$ *for any* $\mathcal{E}$ *with* $\emptyset \neq \mathcal{E} \subseteq \mathcal{D}$ *and* $|\mathcal{E}| < \lambda \leq \kappa$.[16]

2. *The following properties of* K_i *are referred to as introspective properties.*
   (a) *Truth Axiom:* $K_i(E) \subseteq E$ *(for any* $E \in \mathcal{D}$*).*
   (b) *Positive Introspection:* $K_i(E) \subseteq K_i K_i(E)$.
   (c) *Negative Introspection:* $(\neg K_i)(E) \subseteq K_i(\neg K_i)(E)$.

3. K_i *satisfies the Kripke property if, for each* $\omega \in \Omega$ *and* $E \in \mathcal{D}$*, the following holds:*

$$\omega \in K_i(E) \text{ if(f) } b_{K_i}(\omega) \subseteq E,$$

where $b_{K_i} : \Omega \to \mathcal{P}(\Omega)$ *is defined as follows. For each* $\omega \in \Omega$,

$$\begin{aligned} b_{K_i}(\omega) &:= \{\omega' \in \Omega \mid \omega' \in E \text{ for all } E \in \mathcal{D} \text{ with } \omega \in K_i(E)\} \\ &= \bigcap \{E \in \mathcal{D} \mid \omega \in K_i(E)\}. \end{aligned}$$

No-Contradiction Axiom means that a player cannot know any contradiction, which is represented as the empty set. Consistency means that, if a player knows an event E then she also consider E possible in the sense that she does not know its negation E^c. In other words, she cannot know E and E^c at the same time. This notion of possibility is considered to be the dual notion of knowledge (e.g., Hintikka [43]). Thus, we define the possibility operator $L_{K_i} : \mathcal{D} \to \mathcal{D}$ by $L_{K_i}(E) := (\neg K_i)(E^c)$. We say that a player i considers an event $E \in \mathcal{D}$ possible at a state $\omega \in \Omega$ if $\omega \in L_{K_i}(E)$. Now, Consistency means that knowledge implies possibility.

Monotonicity says that if a player knows some event then she knows any of its logical consequences. Necessitation means that a player knows any form of tautology such as $E \cup E^c$, where note that any such tautology is expressed as Ω.

Non-empty (λ-)Conjunction implies that a player knows any non-empty conjunction of events (with cardinality less than λ) if she knows each event. We refer to Non-empty $\aleph_0$- ($\aleph_1$-)Conjunction as Non-empty Finite (Countable) Conjunction. Note that Non-empty (λ-)Conjunction and Necessitation is jointly equivalent to (λ-)Conjunction, since $\Omega = \bigcap \emptyset$ (i.e., Necessitation can be seen as the empty conjunction property).

Truth Axiom says that a player can only know what is true. Truth Axiom distinguishes knowledge from belief in the sense that belief can be false while knowledge has to be true. However, we generically refer to such (qualitative) belief as knowledge (without Truth Axiom). Positive Introspection states that if a player knows some

[16] We make two remarks. First, ∞-Conjunction means Arbitrary Conjunction. Second, for a given infinite (regular) cardinal κ or $\kappa = \infty$, we only consider λ-Conjunction with $\lambda \leq \kappa$. We could denote (κ, λ)-Conjunction if we need to emphasize that the domain $\mathcal{D}$ is a κ-complete algebra.

event then she knows that she knows it. Negative Introspection states that if a player does not know some event then she knows that she does not know it.

Next, we examine what the Kripke property means. To that end, we define a notion of possibility between states. For states ω and ω' in Ω, we say that ω' is considered *possible* by i at ω if $\omega' \in b_{K_i}(\omega)$. In words, ω' is considered possible at ω iff for any event $E \in \mathcal{D}$ which player i knows at ω, the event E is true at ω'. Thus, $b_{K_i}(\omega)$ induces a binary relation (a possibility relation) in the sense that $b_{K_i}(\omega)$ comprises exactly of the set of states considered possible by i at ω. Note, however, that $b_{K_i}(\omega)$ may not necessarily be an event when $\mathcal{D}$ is not a complete algebra. Now, the Kripke property means that i knows an event E at a state ω iff E contains any state considered possible by i at ω.

We make three remarks on the definition of the possibility relation between states as well as the Kripke property. First, the logical and introspective properties of K_i can be stated as the corresponding properties of b_{K_i}. We will revisit some instances in Proposition 12 in Section 5.2. For example, under the Kripke property, K_i satisfies Truth Axiom, Positive Introspection, and Negative Introspection iff b_{K_i} yields a partition on Ω.

Second, the Kripke property implies the following form of conjunction property. For any subset $\mathcal{E}$ of $\mathcal{D}$ such that $\bigcap \mathcal{E} \in \mathcal{D}$ and $\omega \in K_i(E)$ for all $E \in \mathcal{E}$, we have $\omega \in K_i(\bigcap \mathcal{E})$. Thus, the Kripke property implies κ-Conjunction (including Necessitation) as well as Monotonicity. The converse, however, is not necessarily true unless $\mathcal{D}$ is a complete algebra.[17]

Third, it would be natural to ask how the possibility in terms of $\omega' \in b_{K_i}(\omega)$ relates to the possibility in the sense of $\omega \in L_{K_i}(\{\omega'\})$. For example, Morris [64] introduces a possibility correspondence from the possibility operator L_{K_i}. If every singleton set $\{\omega'\}$ is an event and if an i's knowledge satisfies Monotonicity, then b_{K_i} can be written as

$$b_{K_i}(\omega) = \{\omega' \in \Omega \mid \omega \in L_{K_i}(\{\omega'\})\}. \tag{1}$$

For completeness, the proof is in Remark A.1 in the Appendix.

In this section, so far, we have defined the notion of knowledge spaces (Definition 1) and the properties of knowledge operators (Definition 2). When we speak of a knowledge space, it resides in a given class of knowledge spaces satisfying given assumptions on players' knowledge. Note that we can assume different properties of knowledge on different players.[18] Note also that some property of knowledge implies

[17] Samet [78] provides specific examples where a knowledge operator on an ($\aleph_0$-complete) algebra satisfies all the logical and introspective properties (i.e., "S5" knowledge, where Finite Conjunction is considered) but is not derived from a partition (i.e., fails to satisfy the Kripke property). Proposition A.1 in the Appendix provides a characterization of the Kripke property. This generalizes Samet's [78] condition under which "S5" knowledge is derived from a partition.

[18] For example, we can accommodate a case where players have multiple epistemic operators. Suppose that player i's knowledge operator (which satisfies Truth Axiom) is given by $K_{(0,i)}$ while her (non-probabilistic) belief operator (which may violate Truth Axiom) is given by $K_{(1,i)}$. We can carry out our analysis by regarding the set of players as $\{0,1\} \times I$.

another.[19]

2.3 Definition of a Universal Knowledge Space

Our main objective of the paper (Theorem 1 in Section 3) is to demonstrate the existence of a universal κ-knowledge space for any infinite (regular) cardinal κ and for any combination of assumptions on players' knowledge. To that end, we define a universal knowledge space (in a given class of knowledge spaces). It is a knowledge space to which every knowledge space (in the given class) is uniquely mapped in a knowledge-preserving manner. We start with formalizing the notion of a mapping that preserves states of nature and players' knowledge, i.e., the notion of a knowledge morphism between knowledge spaces.

Definition 3 (Knowledge Morphism). *Let* $\overrightarrow{\Omega} := \langle(\Omega, \mathcal{D}), (K_i)_{i \in I}, \Theta\rangle$ *and* $\overrightarrow{\Omega'} := \langle(\Omega', \mathcal{D}'), (K'_i)_{i \in I}, \Theta'\rangle$ *be knowledge spaces of a given class. A knowledge morphism* $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ *is a mapping* $\varphi : \Omega \to \Omega'$ *with the following three properties.*

1. *For all* $E' \in \mathcal{D}'$, $\varphi^{-1}(E') \in \mathcal{D}$.
2. *For all* $\omega \in \Omega$, $\Theta'(\varphi(\omega)) = \Theta(\omega)$.
3. *For each* $i \in I$ *and* $E' \in \mathcal{D}'$, $K_i(\varphi^{-1}(E')) = \varphi^{-1}(K'_i(E'))$.

Condition (1) means that $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ is κ-measurable. This is a version of the measurability condition imposed on a type morphism in the type space literature. This condition ensures that the inverse map $\varphi^{-1} : \mathcal{D}' \to \mathcal{D}$ embed set-algebraic operations equipped within $\mathcal{D}'$ into $\mathcal{D}$. It is to be noted that we do not require $\varphi(\mathcal{D}) \subseteq \mathcal{D}'$.

Condition (2) requires that the same state of nature prevail for two associated knowledge spaces. Condition (3) requires that players' knowledge be preserved from one space to another in the following sense: for any event E' in $\overrightarrow{\Omega'}$, player i knows E' at $\varphi(\omega)$ iff she knows $\varphi^{-1}(E')$ at ω.

Before we formally define the notion of a universal knowledge space, we examine the concept of knowledge morphism further. First, for any given knowledge space $\overrightarrow{\Omega}$, the identity map $\mathrm{id}_\Omega : \Omega \to \Omega$ is a knowledge morphism from $\overrightarrow{\Omega}$ into itself. We denote by $\mathrm{id}_{\overrightarrow{\Omega}} : \overrightarrow{\Omega} \to \overrightarrow{\Omega}$ the identity knowledge morphism.

Second, we call a knowledge morphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ a *knowledge isomorphism*, if there is a knowledge morphism $\psi : \overrightarrow{\Omega'} \to \overrightarrow{\Omega}$ such that $\psi \circ \varphi = \mathrm{id}_{\overrightarrow{\Omega}}$ and $\varphi \circ \psi = \mathrm{id}_{\overrightarrow{\Omega'}}$.

[19] For example, Truth Axiom implies No-Contradiction Axiom and Consistency. The combination of Consistency and Necessitation (that of Consistency and Finite Conjunction) also imply No-Contradiction Axiom. Regarding this point, Negative Introspection is somewhat strong in the following sense. Negative Introspection together with Truth Axiom imply Positive Introspection (see, for example, Aumann [5, p. 270]). Also, Negative Introspection, Monotonicity, and Truth Axiom imply κ-Conjunction including Necessitation (Corollary 2 in Section 4.1).

In other words, a knowledge morphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a knowledge isomorphism if φ is bijective and its inverse φ^{-1} is a knowledge morphism. Note that if φ is a knowledge isomorphism then its inverse φ^{-1} is unique. We say that knowledge spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$ are isomorphic, if there is a knowledge isomorphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$.

Third, we define the notion of a knowledge subspace in an analogous way to the notion of a belief subspace in the type space literature (Mertens and Zamir [59, Definition 2.15]).

Definition 4 (Knowledge Subspace)**.** *Let $\overrightarrow{\Omega} := \langle(\Omega, \mathcal{D}), (K_i)_{i\in I}, \Theta\rangle$ be a knowledge space. We call a knowledge space $\overrightarrow{\Omega'} = \langle(\Omega', \mathcal{D}'), (K'_i)_{i\in I}, \Theta'\rangle$ a knowledge subspace (of $\overrightarrow{\Omega}$) if $\Omega' \subseteq \Omega$ and if the inclusion map $i_{\Omega',\Omega} : \Omega' \to \Omega$ is a knowledge morphism. If a knowledge subspace $\overrightarrow{\Omega'}$ additionally satisfies that $K'_i(\Omega') = \Omega'$ for all $i \in I$, then we call $\overrightarrow{\Omega'}$ a knowledge closed subspace.*[20]

Now, we define a universal knowledge space. We formalize below the idea that a universal knowledge space "contains" all knowledge spaces in the sense that any knowledge space can be mapped uniquely to the universal space by a knowledge morphism.

Definition 5 (Universal Knowledge Space)**.** *Fix a class of (κ-)knowledge spaces (of I on $(S, \mathcal{A}_S)$). A knowledge space $\overrightarrow{\Omega^*}$ is said to be universal if, for any knowledge space $\overrightarrow{\Omega}$, there is a unique knowledge morphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$.*

We make the following remarks. Fix an infinite cardinal κ (or $\kappa = \infty$), a non-empty set of players I, a κ-complete algebra of states of nature $(S, \mathcal{A}_S)$, and assumptions on players' knowledge. The following can be verified: (i) any composite of two knowledge morphisms is a knowledge morphism; (ii) composites of knowledge morphisms are associative; and (iii) the identity mapping is a knowledge morphism satisfying the identity law. This means that the collection of all κ-knowledge spaces of I on $(S, \mathcal{A}_S)$ forms a category, where each knowledge space $\overrightarrow{\Omega}$ is an *object* and a knowledge morphism is a *morphism*.[21]

In the language of category theory, a universal knowledge space is a *terminal (final)* object in the category of knowledge spaces.[22] As it is well known in category theory that a terminal object is unique up to isomorphism, a universal knowledge space is unique up to knowledge isomorphism.

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[20] Note that we call such $\overrightarrow{\Omega'}$ a knowledge closed subspace even though the original knowledge space $\overrightarrow{\Omega}$ fails to satisfy Necessitation.

[21] See also Remark A.2 in the Appendix for additional remarks on a class of knowledge spaces as a category.

[22] A knowledge space satisfying Definition 5 can be called a terminal knowledge space in line with the language of category theory (see also Brandenburger and Keisler [18, Section 11]). We, however, simply call such a space to be a universal space.

In the following, we assume that S is not empty. For the degenerate case where $(S, \mathcal{A}_S) = (\emptyset, \{\emptyset\})$, the class of knowledge spaces of I on $(\emptyset, \{\emptyset\})$ consists of the trivial knowledge space $\overrightarrow{\emptyset} = \langle(\emptyset, \{\emptyset\}), (K_i)_{i\in I}, \Theta\rangle$, where $K_i = \mathrm{id}_{\{\emptyset\}}$ for all $i \in I$ and Θ is the empty function. This follows because there is no mapping from a non-empty set Ω into the empty set S. Thus, the category of knowledge spaces of I on $(\emptyset, \{\emptyset\})$ is the category consisting of a single object and its identity morphism, and the single object is terminal. In the language of category theory, the trivial knowledge space $\overrightarrow{\emptyset}$ is an initial knowledge space in a given category of knowledge spaces on any $(S, \mathcal{A}_S)$.

3 A Syntactic Approach to a Universal Knowledge Space

Throughout this section, fix a non-empty set of players I, an infinite regular cardinal κ, and a κ-complete algebra $(S, \mathcal{A}_S)$ of states of nature. We also fix assumptions on players' knowledge. Thus, any knowledge space refers to a κ-knowledge space of I on $(S, \mathcal{A}_S)$ in the given category.

3.1 Syntactic Construction of a Universal Knowledge Space

We demonstrate the existence of a universal (κ-)knowledge space by employing the "expressions-descriptions" approach (HS [39] and Meier [55, 56]).[23] We do so without imposing any restriction on a κ-complete algebra $(S, \mathcal{A}_S)$.[24]

In the following, we take six steps in order to establish the existence of a universal knowledge space. The first step is to inductively define *expressions*. Expressions are syntactic formulas that express events defined solely in terms of nature and players' knowledge. Specifically, any event of nature $E \in \mathcal{A}_S$ is an object of interactive knowledge, so that any such E is an expression. Since objects of knowledge are closed under κ-union, κ-intersection, and complementation, we define the corresponding syntactic operations. Also, each player's knowledge is an object of interactive knowledge, and thus we define expressions denoting players' knowledge.[25]

Definition 6 (Expressions: Logical Formulas Expressing Nature and Interactive Knowledge)**.** *The set of all (κ-)expressions $\mathcal{L}(= \mathcal{L}^I_\kappa(\mathcal{A}_S))$ is the smallest set which satisfies the following.*

[23] See also Meier [57] and Moss and Viglizzo [65, 66] for related developments of this approach.

[24] This means that the constructions in Meier [55, 56] could be generalized. In Meier [55, 56], the following regularity ("separative") condition on $(S, \mathcal{A}_S)$ is imposed: for any distinct $s, s' \in S$, there is $E \in \mathcal{A}_S$ with $s \in E$ and $s' \notin E$. In other words, $\{s\} = \bigcap\{E \in \mathcal{A}_S \mid s \in E\}$ for each $s \in S$, but it may be the case that $\{s\} \notin \mathcal{A}_S$ as $\mathcal{A}_S$ is not necessarily closed under arbitrary intersection.

[25] Thus, expressions are infinitary langauages. Fagin [25] and Heifetz [37] analyze their infinitary epistemic logics using such infinitary languages.

1. *Every $E \in \mathcal{A}_S$ is an expression.*

2. *If $\mathcal{E}$ is a set of expressions with $|\mathcal{E}| < \kappa$, then $(\bigwedge \mathcal{E})$ is an expression, with the following conventions. First, we let $S := \bigwedge \emptyset$. Second, we identify $\bigwedge \mathcal{E} := \bigcap \mathcal{E}$ if $\mathcal{E}$ is a subset of $\mathcal{A}_S$ with $|\mathcal{E}| < \kappa$.*

3. *If e is an expression then $(\neg e)$ is an expression, where we identify $(\neg E) := E^c$ for all $E \in \mathcal{A}_S$.*

4. *If e is an expression, then $(k_i(e))$ is an expression for each $i \in I$.*

For ease of notation, we often add or omit parentheses in denoting expressions. We also introduce other notations. First, if $\mathcal{E}$ is a set of expressions with $|\mathcal{E}| < \kappa$, then we let $(\bigvee \mathcal{E}) := \neg(\bigwedge\{(\neg e) \in \mathcal{L} \mid e \in \mathcal{E}\})$ with the convention that $\bigvee \emptyset := \emptyset$. Second, we interchangeably denote, for instance, $e_1 \wedge e_2 = \bigwedge\{e_1, e_2\}$ and $e_1 \vee e_2 = \bigvee\{e_1, e_2\}$.[26] Third, we interchangeably denote $\bigwedge_{j\in J} e_j = \bigwedge\{e_j \mid j \in J\}$ and $\bigvee_{j\in J} e_j = \bigvee\{e_j \mid j \in J\}$ when the set of expressions are given by some index set J. Fourth, we introduce $(e \to f) := ((\neg e) \vee f)$ and $(e \leftrightarrow f) := ((e \to f) \wedge (f \to e))$.

We remark that the set $\mathcal{L}$ incorporates all the hierarchies of interactive knowledge regarding $(S, \mathcal{A}_S)$ up to the ordinality of κ. To see this point, we see how the set of expressions $\mathcal{L}$ is inductively generated from the set of states of nature $(S, \mathcal{A}_S)$ in κ steps.

Remark 1 (Restatement of Expressions $\mathcal{L}$)**.** We let $\mathcal{L}_0 := \mathcal{A}_S$. For any ordinal α with $0 < \alpha \leq \kappa$, we define

$$\mathcal{L}'_\alpha := \left(\bigcup_{\beta<\alpha} \mathcal{L}_\beta\right) \cup \bigcup_{i\in I}\{(k_i(e)) \mid e \in \bigcup_{\beta<\alpha} \mathcal{L}_\beta\}; \text{ and}$$
$$\mathcal{L}_\alpha := \mathcal{L}'_\alpha \cup \{(\neg e) \mid e \in \mathcal{L}'_\alpha\} \cup \left\{\bigwedge \mathcal{F} \,\middle|\, \mathcal{F} \subseteq \mathcal{L}'_\alpha \text{ and } 0 < |\mathcal{F}| < \kappa\right\}.$$

Then, we have $\mathcal{L} = \mathcal{L}_\kappa$.

Intuitively, each expression $e \in \mathcal{L}_\alpha$ is an expression of "depth at most α." Thus, Remark 1 states that the set $\mathcal{L}$ consists exactly of expressions of "depth at most κ," i.e., logical formulas expressing interactive knowledge regarding $(S, \mathcal{A}_S)$ up to the ordinality of κ.

While expressions themselves are defined independently of any particular knowledge space, for any given knowledge space $\overrightarrow{\Omega}$, we can recursively identify each expression with an event in $\overrightarrow{\Omega}$ (i.e., an element of $\mathcal{D}$), by the measurability condition imposed on the knowledge space (i.e., $\Theta^{-1}(\mathcal{A}_S) \subseteq \mathcal{D}$).

[26] Thus, for example, we simply do not distinguish $e_1 \vee e_2$ and $e_2 \vee e_1$. Similarly, since $\{e, e\} = \{e\}$, we simply identify $(e \wedge e)$ as e. These could be augmented by defining $(\bigwedge \mathcal{F})$ for an ordinal sequence of expressions $\mathcal{F}$ instead of a set of expressions.

Definition 7 (Expressions are Identified as Events). *Fix a (κ-)knowledge space $\overrightarrow{\Omega}$. We inductively define the mapping $[\![\cdot]\!]_{\overrightarrow{\Omega}} : \mathcal{L} \to \mathcal{D}$, which we call the semantic interpretation function of $\overrightarrow{\Omega}$, as follows.*

1. $[\![E]\!]_{\overrightarrow{\Omega}} := \Theta^{-1}(E)$ *for every* $E \in \mathcal{A}_S$.
2. $[\![\bigwedge \mathcal{E}]\!]_{\overrightarrow{\Omega}} := \bigcap_{e \in \mathcal{E}} [\![e]\!]_{\overrightarrow{\Omega}}$ *for any* $\mathcal{E}$ *with* $\mathcal{E} \subseteq \mathcal{L}$ *and* $|\mathcal{E}| < \kappa$.
3. $[\![\neg e]\!]_{\overrightarrow{\Omega}} := \neg [\![e]\!]_{\overrightarrow{\Omega}} \left(= ([\![e]\!]_{\overrightarrow{\Omega}})^c\right)$ *for each expression* e.
4. $[\![k_i(e)]\!]_{\overrightarrow{\Omega}} := K_i([\![e]\!]_{\overrightarrow{\Omega}})$ *for each* $i \in I$ *and expression* e.

We call $[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D}$ *the denotation of* e *in* $\overrightarrow{\Omega}$.

Note that the semantic interpretation function of a given knowledge space is, by recursion, uniquely extended from Θ^{-1}. It gives the semantic meaning of an expression e in the very sense that $[\![e]\!]_{\overrightarrow{\Omega}}$ is the set of states of the world in which the expression e holds. Again, we often add or omit parentheses for ease of notation. Now, we establish that a knowledge morphism preserves semantics.

Lemma 1 (Knowledge Morphism Preserves Semantics/Meanings of Expressions). *If* $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ *is a knowledge morphism between knowledge spaces* $\overrightarrow{\Omega}$ *and* $\overrightarrow{\Omega'}$*, then* $[\![\cdot]\!]_{\overrightarrow{\Omega}} = \varphi^{-1}([\![\cdot]\!]_{\overrightarrow{\Omega'}})$.

Before we go to the second step, we define the following semantic notions and see how a knowledge morphism preserves these notions. Our main purpose is to characterize these notions in a universal knowledge space.

Definition 8 (Semantic Properties).

1. *We say that an expression* $e \in \mathcal{L}$ *is valid in a knowledge space* $\overrightarrow{\Omega}$ *(written* $\models_{\overrightarrow{\Omega}} e$*) if* $[\![e]\!]_{\overrightarrow{\Omega}} = \Omega$. *If* e *is valid in any knowledge space (of the given category), then we say that* e *is valid (written* $\models e$*) (in the given category).*
2. *A set of expressions* $\Phi (\subseteq \mathcal{L})$ *is said to be satisfiable in* $\overrightarrow{\Omega}$ *if there is a state* $\omega \in \Omega$ *such that* $\omega \in [\![f]\!]_{\overrightarrow{\Omega}}$ *for all* $f \in \Phi$. *If there is a knowledge space* $\overrightarrow{\Omega}$ *such that* Φ *is satisfiable, then we say that* Φ *is satisfiable.*
3. *We say that* $e \in \mathcal{L}$ *is a semantic consequence of* Φ *in* $\overrightarrow{\Omega}$ *(written* $\Phi \models_{\overrightarrow{\Omega}} e$*) if,* $\omega \in [\![e]\!]_{\overrightarrow{\Omega}}$ *holds whenever* $\omega \in [\![f]\!]_{\overrightarrow{\Omega}}$ *for all* $f \in \Phi$. *We say that* $e \in \mathcal{L}$ *is a semantic consequence of* Φ *(written* $\Phi \models e$*) if* $\Phi \models_{\overrightarrow{\Omega}} e$ *for any knowledge space* $\overrightarrow{\Omega}$.

We have four remarks regarding the notion of validity. First, irrespective of assumptions on players' knowledge, such expressions as S and $(e \vee (\neg e))$ are valid. Second, in a category of knowledge spaces where, for example, Truth Axiom is always assumed on player i, an expression of the form $(k_i(e) \to e)$ is always valid. On the other hand, in a category of knowledge spaces where Truth Axiom is not necessarily assumed on player i, an expression of the form $(k_i(e) \to e)$ is not necessarily valid. Third, it is generally possible that a given expression f happens to be valid in some knowledge space $\overrightarrow{\Omega}$ of a given category due to a particular representation of states (i.e., in a particular context) and players' knowledge while it is not a valid expression in other knowledge spaces of the same category.

Fourth, Lemma 1 implies that any valid expression e in $\overrightarrow{\Omega'}$ is also valid in $\overrightarrow{\Omega}$. This is because $[\![e]\!]_{\overrightarrow{\Omega}} = \varphi^{-1}([\![e]\!]_{\overrightarrow{\Omega'}}) = \varphi^{-1}(\Omega') = \Omega$. If Φ is satisfiable in $\overrightarrow{\Omega}$, then so is it in $\overrightarrow{\Omega'}$. Also, suppose that $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a surjective knowledge morphism. If e is a semantic consequence of Φ in $\overrightarrow{\Omega}$, then so is it in $\overrightarrow{\Omega'}$.

The second step is to define *descriptions* by the set of expressions that obtain at each state of the world in each given knowledge space, together with the corresponding state of nature. Since states of nature and expressions reside in different spaces, we define a description to be a subset of the disjoint union $S \sqcup \mathcal{L}(:= \{(0, s) \in \{0\} \times S \mid s \in S\} \cup \{(1, e) \in \{1\} \times \mathcal{L} \mid e \in \mathcal{L}\})$ as follows.[27]

Definition 9 (Description: Set of Expressions Prevailing at Some State of Some Knowledge Space)**.** *For any given knowledge space $\overrightarrow{\Omega}$ and $\omega \in \Omega$, we define $D(\omega)$, which we call the description of ω, by*

$$\begin{aligned} D(\omega) &:= \{\Theta(\omega)\} \sqcup \{e \in \mathcal{L} \mid \omega \in [\![e]\!]_{\overrightarrow{\Omega}}\} \\ &:= \{(0, \Theta(\omega))\} \cup \{(1, e) \in \{1\} \times \mathcal{L} \mid \omega \in [\![e]\!]_{\overrightarrow{\Omega}}\}. \end{aligned}$$

Note that $(0, s) \in D(\omega)$ indicates which event of nature belongs to $D(\omega)$ in the following sense: for any $E \in \mathcal{A}_S$, we have $(1, E) \in D(\omega)$ iff $s \in E$. We, however, simply keep track of every expression e which is true at ω in the description $D(\omega)$.

Descriptions have two roles in establishing the existence of a universal knowledge space. First, we will construct a universal knowledge space in such a way that its underlying set Ω^* of states of the world is the set of all descriptions of states of the world ranged over all knowledge spaces (in the given category). Thus, we define Ω^* as follows:

$$\Omega^* := \{\omega^* \in \mathcal{P}(S \sqcup \mathcal{L}) \mid \omega^* = D(\omega) \text{ for some } \overrightarrow{\Omega} \text{ and } \omega \in \Omega\}. \tag{2}$$

By definition, the set Ω^* is not empty as long as there is a knowledge space $\overrightarrow{\Omega}$ with $\Omega \neq \emptyset$ in the given category. For any assumptions on players' knowledge, we can simply construct a knowledge space $\overrightarrow{\{s\}} := \langle(\{s\}, \mathcal{P}(\{s\})), (\mathrm{id}_{\mathcal{P}(\{s\})})_{i \in I}, i_{\{s\},S}\rangle$ where $s \in S$. It can be seen that each $K_i = \mathrm{id}_{\mathcal{P}(\{s\})}$ satisfies all the properties of knowledge. Thus, Ω^* is indeed not empty.

[27] Throughout the paper, we keep denoting by $(0, s)$ and $(1, e)$ elements of $S \sqcup \mathcal{L}$.

Note also that the set Ω^* depends on the choice of a class of knowledge spaces. Consider any two classes of knowledge spaces where the set of assumptions on players' knowledge in the first class is a subset of that in the second class. Denote by Ω^{1*} and Ω^{2*} the spaces constructed according to Equation (2). Then, we have $\Omega^{2*} \subseteq \Omega^{1*}$. This follows because if $\omega^* \in \Omega^{2*}$ then there are a knowledge space $\overrightarrow{\Omega}$ (in the second class) and a state $\omega \in \Omega$ such that $\omega^* = D(\omega)$. Since $\overrightarrow{\Omega}$ is also an object in the first class, we get $\omega^* = D(\omega) \in \Omega^{1*}$.

Second, we regard D as a mapping $D : \Omega \to \Omega^*$ for any given knowledge space $\overrightarrow{\Omega}$, and hence we call $D : \Omega \to \Omega^*$ to be the *description map*. We denote the description map by $D_{\overrightarrow{\Omega}} : \Omega \to \Omega^*$ when we stress the domain of D. The description map D will turn out to be a knowledge morphism between $\overrightarrow{\Omega}$ and our candidate universal knowledge space "$\overrightarrow{\Omega^*}$."

Before we go to the next step, we show that it follows from Lemma 1 that a knowledge morphism preserves the descriptions. To that end, observe the following. Fix knowledge spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$. For any $(\omega, \omega') \in \Omega \times \Omega'$, we have $D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega'}}(\omega')$ iff the following hold: (i) $\Theta(\omega) = \Theta'(\omega')$; and (ii) $\omega \in [\![e]\!]_{\overrightarrow{\Omega}}$ iff $\omega' \in [\![e]\!]_{\overrightarrow{\Omega'}}$ for all $e \in \mathcal{L}$. Thus, $D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega'}}(\omega')$ means that the outside analysts would consider states ω and $\varphi(\omega)$ to be equivalent in terms of a prevailing state of nature and prevailing expressions, abstracting away from physical representations of $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$.[28]

Both conditions are met for any $(\omega, \varphi(\omega)) \in \Omega \times \Omega'$ such that $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a knowledge morphism. Thus, a knowledge morphism preserves the descriptions.

Corollary 1 (Knowledge Morphism Preserves Descriptions)**.** *Let $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ be a knowledge morphism between knowledge spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$. Then, $D_{\overrightarrow{\Omega}} = D_{\overrightarrow{\Omega'}} \circ \varphi$.*

We make the following two remarks on Corollary 1. First, we say that a knowledge space $\overrightarrow{\Omega}$ is *non-redundant* (Mertens and Zamir [59, Definition 2.4]) if its description map D is injective. In other words, for any distinct ω and ω', either $\Theta(\omega) \neq \Theta(\omega')$ or they are separated by (a κ-complete sub-algebra) $\mathcal{D}_\kappa := \{[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid e \in \mathcal{L}\}$.[29] Note that, in Section 3.3, we will represent $\mathcal{D}_\kappa$ in terms of the primitives of the knowledge space $\overrightarrow{\Omega}$ alone (i.e., the notion of non-redundancy can be defined in terms of the primitives of a given knowledge space).

[28]This notion of equivalence (identicalness, or indistingushability) is closely related to the following: Fagin [25, Section 4] in the context of epistemic logic and Mertens and Zamir [59] in terms of belief hierarchies in the universal type space literature. Also, as it turns out that the description map is a unique knowledge morphism into a universal knowledge space, this notion of equivalence corresponds to one notion of bisimulations called "behavioral equivalence" (Kurz [46]) in the literature of category theory, computer science, and logic. We discuss this point in Remark A.3 in the Appendix. For notions of bisimulations ("observational equivalence"), see, for instance, Jacobs and Rutten [44], Kurz [46], Rutten [73], and the references therein.

[29]In Fagin's [25] terminology, such a knowledge space is said to be non-flabby.

Second, Corollary 1 implies that if $\overrightarrow{\Omega'}$ is non-redundant then there is at most one knowledge morphism from a given space $\overrightarrow{\Omega}$ into $\overrightarrow{\Omega'}$.[30] If $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ and $\psi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ are knowledge morphisms then $D_{\overrightarrow{\Omega'}} \circ \varphi = D_{\overrightarrow{\Omega}} = D_{\overrightarrow{\Omega'}} \circ \psi$. Since $D_{\overrightarrow{\Omega'}}$ is injective, it follows that $\varphi = \psi$. We will show that the description map $D_{\overrightarrow{\Omega}} : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ is a unique knowledge morphism by demonstrating that $D_{\overrightarrow{\Omega^*}} : \overrightarrow{\Omega^*} \to \overrightarrow{\Omega^*}$ is the identity map.

The third step is to define the collection of events on Ω^* (i.e., the domain of our candidate universal knowledge space). Since each expression e corresponds to an object of knowledge, we define the set of descriptions $[e]$ that make e true (i.e., the set of descriptions that contain e) to be objects of players' knowledge on Ω^*.

Formally, for each $e \in \mathcal{L}$, we define the set of descriptions $[e]$ by $[e] := \{\omega^* \in \Omega^* \mid (1, e) \in \omega^*\}$. We then define the collection $\mathcal{D}^* := \{[e] \in \mathcal{P}(\Omega^*) \mid e \in \mathcal{L}\}$ on Ω^*. Now, we show that $\mathcal{D}^*$ is a legitimate domain (i.e., a κ-complete algebra).

Lemma 2 (Domain of Candidate Universal Space)**.** *$(\Omega^*, \mathcal{D}^*)$ is a κ-complete algebra. Moreover, for any knowledge space $\overrightarrow{\Omega}$, the description map $D : (\Omega, \mathcal{D}) \to (\Omega^*, \mathcal{D}^*)$ is a κ-measurable mapping such that $D^{-1}([e]) = [\![e]\!]_{\overrightarrow{\Omega}}$ for any $e \in \mathcal{L}$.*

The property that $D^{-1}_{\overrightarrow{\Omega}}([\cdot]) = [\![\cdot]\!]_{\overrightarrow{\Omega}}$ exhibits a duality between the semantic interpretation function and the description map in the following sense. By recursion, the semantic interpretation function $[\![\cdot]\!]_{\overrightarrow{\Omega}}$ is a unique map from the set of expressions $\mathcal{L}$ into the domain $\mathcal{D}$ of a given knowledge space. On the other hand, the description map $D_{\overrightarrow{\Omega}}$ is going to be a unique map from the underlying states Ω into the set of descriptions Ω^*.

Note also that the κ-complete sub-algebra $\mathcal{D}_\kappa$ is the one induced by $D_{\overrightarrow{\Omega}}$ in the sense that $\mathcal{D}_\kappa = \{D^{-1}_{\overrightarrow{\Omega}}([e]) \in \mathcal{D} \mid e \in \mathcal{L}\}$.

The fourth step is to introduce players' knowledge. We define each player's knowledge regarding $\mathcal{D}^*$ in a way that an agent i knows an event $[e]$ at a state ω^* iff ω^* contains $k_i(e)$ (i.e., $(1, k_i(e)) \in \omega^*$). We show that this is well defined: for any expressions e and f, if they are equivalent in the sense that $(1, e) \in \omega^*$ iff $(1, f) \in \omega^*$, then $k_i(e)$ and $k_i(f)$ are equivalent in the same sense.[31] This equivalence depends on the imposed assumptions on players' knowledge. For example, if Positive Introspection and Truth Axiom are imposed on player i, then $k_i(e)$ and $k_i k_i(e)$ are equivalent in

[30] An object having this property is called extensional in Kurz [46]. This is also related to a notion of "coinduction proof principle" in the literature of category theory, computer science, and logic (see, for example, Jacobs and Rutten [44], Kurz [46], Rutten [73], and the references therein) in the following sense. If $\overrightarrow{\Omega}$ is non-redundant, then in order for two states ω and ω' in Ω to be the same, it is enough to show that $D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega}}(\omega')$ (i.e., they are behaviorally equivalent).

[31] First, this equivalence is, in spirit, related to the inference rule of equivalence in a syntactic system (see, for example, Lismont and Mongin [50]) and the notion which Dekel, Lipman, and Rustichini (DLR) [21] calls event sufficiency. Second, we will discuss in Proposition 2 that if there are a knowledge space $\overrightarrow{\Omega}$ and a state $\omega \in \Omega$ such that $[\![e]\!]_{\overrightarrow{\Omega}} \neq [\![f]\!]_{\overrightarrow{\Omega}}$ then e and f are not equivalent (i.e., $[e] \neq [f]$). Thus, such identifications (equivalences) of expressions are minimal in $\overrightarrow{\Omega^*}$.

the above sense. We will examine how assumptions on each player's knowledge are encoded within Ω^* itself.

Lemma 3 (Knowledege Operators of Candidate Universal Space)**.** *Fix $i \in I$. We define $K_i^* : \mathcal{D}^* \to \mathcal{D}^*$ by $K_i^*([e]) := [k_i(e)]$ for each $e \in \mathcal{L}$. Then, K_i^* is a well-defined knowledge operator which inherits the properties of knowledge imposed in the given category. Moreover, for any given knowledge space $\overrightarrow{\Omega}$, $D^{-1}(K_i^*([e])) = K_i(D^{-1}([e]))$ for all $[e] \in \mathcal{D}^*$.*

In the Appendix, we provide the following two results. First, we show in Lemma A.1 that each K_i^* can also inherit other potential properties satisfied in the given category.[32] Second, we show in Proposition A.2 that if there is a knowledge space $\overrightarrow{\Omega}$ which fails to satisfy a given property with respect to some event $[\![e]\!]_{\overrightarrow{\Omega}}$, then the knowledge operator K_i^* fails to satisfy that property with respect to $[e]$.

The fifth step is to construct the mapping $\Theta^* : \Omega^* \to S$ that associates with each state of the world $\omega^* \in \Omega^*$ a state of nature in the following way. For each $\omega^* \in \Omega^*$, we extract the unique state of nature s contained in ω^*.

Lemma 4 (Mapping of Candidate Universal Space)**.** *There is a κ-measurable mapping $\Theta^* : (\Omega^*, \mathcal{D}^*) \to (S, \mathcal{A}_S)$ with the following two properties: (i) $\Theta^*(D(\omega)) = \Theta(\omega)$ for any knowledge space $\overrightarrow{\Omega}$ and $\omega \in \Omega$; and (ii) $(\Theta^*)^{-1}(E) = [E] \in \mathcal{D}^*$ for all $E \in \mathcal{A}_S$.*

So far, we have established the following: (i) $\overrightarrow{\Omega^*} = \langle (\Omega^*, \mathcal{D}^*), (K_i^*)_{i \in I}, \Theta^* \rangle$ is a legitimate knowledge space in the given category of knowledge spaces; and (ii) for any knowledge space $\overrightarrow{\Omega}$, the description map $D : \Omega \to \Omega^*$ is a knowledge morphism.

Before we go on to the next (final) step, we examine how the knowledge space $\overrightarrow{\Omega^*}$ resolves the following form of self-reference: generally, each player's knowledge is defined on states while states are supposed to be complete descriptions of the world.[33] Recall that each player's knowledge at each state ω^* is built within the state ω^* itself in the sense that $\omega^* \in K_i^*([e])$ iff $(1, k_i(e)) \in \omega^*$. The following proposition shows how each player's knowledge is encoded in $\overrightarrow{\Omega^*}$.

Proposition 1 (How Knowledge is Encoded within States)**.** *Fix $i \in I$.*

1. *For each $\omega^* \in \Omega^*$ and $e \in \mathcal{L}$, either $(1, k_i(e)) \in \omega^*$ or $(1, (\neg k_i)(e)) \in \omega^*$.*

2. *For each $\omega^* \in \Omega^*$ and $e \in \mathcal{L}$, at least one of the following holds:*

$$(1, k_i(e)) \in \omega^*, (1, k_i(\neg e)) \in \omega^*, \text{ or } (1, (\neg k_i)(e) \wedge (\neg k_i)(\neg e)) \in \omega^*.$$

   Moreover, i's knowledge satisfies Consistency iff exactly one of them holds for each $\omega^* \in \Omega^*$ and $e \in \mathcal{L}$.

[32] For example, we will show in Section 7 that our arguments apply to richer settings such as dynamic knowledge spaces by using Lemma A.1.

[33] See, for example, Aumann [3, 4, 5], Bacharach [6, 7], Binmore and Brandenburger [11], Brandenburger and Dekel [16], BDG [17], Dekel and Gul [20], FGHV [26], Gilboa [32], Pires [70], Roy and Pacuit [71], Tan and Werlang [82].

3. *For each $\omega^* \in \Omega^*$ and $e \in \mathcal{L}$, exactly one of the following holds:*

$$(1, k_i(e)) \in \omega^*, (1, (\neg k_i)(e) \wedge k_i(\neg k_i)(\neg e)) \in \omega^*, \text{ or } (1, (\neg k_i)(e) \wedge (\neg k_i)(\neg k_i)(e)) \in \omega^*.$$

Moreover, i's knowledge satisfies Negative Introspection iff the third condition never occurs for any $\omega^* \in \Omega^*$ and $e \in \mathcal{L}$.

The first part of Proposition 1 states that, for any $e \in \mathcal{E}$, each state ω^* completely describes i's knowledge of e in the sense that ω^* contains exactly one of the above two expressions denoting (i) i knows e or (ii) i does not know e. The second and third parts characterize how the space Ω^* encodes properties of knowledge (specifically, Consistency and Negative Introspection).[34] We will characterize in Section 3.2 how states encode each property of knowledge.

The sixth step finally establishes that the description map D is a unique knowledge morphism. To that end, we show that the description map from $\overrightarrow{\Omega^*}$ into itself is the identity map on Ω^*.

Lemma 5 (Description Map of Candidate Universal Space)**.** *The description map $D : \Omega^* \to \Omega^*$ is the identity map on Ω^*.*

We consider the following five implications of Lemma 5. The first is that $[\cdot] = [\![\cdot]\!]_{\overrightarrow{\Omega^*}}$. That is, the semantics of e at ω^* is determined by whether $(1, e) \in \omega^*$. Indeed, we prove this property to obtain the result. Second, the knowledge space $\overrightarrow{\Omega^*}$ is non-redundant. The third is the uniqueness of a knowledge morphism $D_{\overrightarrow{\Omega}}$. While the uniqueness follows from the fact that $\overrightarrow{\Omega^*}$ is non-redundant, suppose that $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ is a knowledge morphism. Then we have $D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega^*}}(\varphi(\omega)) = \varphi(\omega)$. Thus, before we go on to the last two implications, we establish our main result, the existence of a universal knowledge space together with the subsequent remark on how the universal knowledge space Ω^* "contains" other knowledge spaces.

Theorem 1 ($\overrightarrow{\Omega^*}$ is Universal)**.** *The space $\overrightarrow{\Omega^*} = \langle (\Omega^*, \mathcal{D}^*), (K_i^*)_{i \in I}, \Theta^* \rangle$ is a universal knowledge space of I on $(S, \mathcal{A}_S)$ for each given category of knowledge spaces.*

Remark 2 (How Universal Space "Contains" Other Knowledge Spaces)**.** Let $\overrightarrow{\Omega}$ be a knowledge space. First, as discussed in Section 2.3, a universal knowledge space exists uniquely up to knowledge isomorphism. It is clear that $\overrightarrow{\Omega}$ is universal iff the description map $D_{\overrightarrow{\Omega}}$ is a knowledge isomorphism.

Second, if $\overrightarrow{\Omega}$ is non-redundant, then, by definition, it is embedded into $\overrightarrow{\Omega^*}$. Generally, there is a knowledge subspace $\overrightarrow{D(\Omega)}$ such that $D : \overrightarrow{\Omega} \to \overrightarrow{D(\Omega)}$ is a (surjective) knowledge morphism. Moreover, if $\overrightarrow{\Omega}$ satisfies Necessitation for every player, then $\overrightarrow{D(\Omega)}$ is a knowledge closed subspace.

[34]Proposition 1 is related to some of Gilboa's [32] consistency conditions for a state to be a complete description of the world.

The fourth implication of Lemma 5 is that, for any state ω of any particular knowledge space $\overrightarrow{\Omega}$, states $\omega \in \Omega$ and $D(\omega) \in \Omega^*$ are equivalent in the sense that the same state of nature $\Theta(\omega) = \Theta^*(D(\omega)) \in S$ prevails and the same set of expressions regarding nature and players' interactive knowledge obtain. This is because $D(\omega) = D_{\overrightarrow{\Omega^*}}(D(\omega))$. To restate, for any representation $\overrightarrow{\Omega}$ of players' interactive knowledge regarding $(S, \mathcal{A}_S)$ and for any realization $\omega \in \Omega$, the prevailing state of nature and the prevailing set of expressions at ω are encoded in the state $D(\omega)$ of the universal knowledge space $\overrightarrow{\Omega^*}$.

The fifth implication of Lemma 5 (indeed, $[\cdot] = [\![\cdot]\!]_{\overrightarrow{\Omega^*}}$) is the following result.

Proposition 2 (Informational Robustness of Universal Space)**.** *Let e be an expression, and let Φ be a set of expressions.*

1. *Φ is satisfiable iff Φ is satisfiable in $\overrightarrow{\Omega^*}$.*
2. *e is a semantic consequence of Φ in every knowledge space iff e is a semantic consequence of Φ in $\overrightarrow{\Omega^*}$.*
3. *e is valid in every knowledge space iff e is valid in $\overrightarrow{\Omega^*}$.*

The first part of Proposition 2 implies that the knowledge space $\overrightarrow{\Omega^*}$ exhausts all possible sets of satisfiable expressions (in some knowledge space $\overrightarrow{\Omega}$) within $\overrightarrow{\Omega^*}$. Put differently, if Φ is a set of expressions that hold at some state ω in some knowledge space $\overrightarrow{\Omega}$, then the expressions in Φ hold at $D(\omega)$ in $\overrightarrow{\Omega^*}$. This result reflects the insight by Moss and Viglizzo [65, 66] that their terminal object (for a "measure polynomial functor") in their category-theoretical version of the expression-description approach consists of all satisfied theories (descriptions) of all points in all objects.

The third part of Proposition 2 implies that if there are expressions e and f and a knowledge space $\overrightarrow{\Omega}$ such that $[\![e]\!]_{\overrightarrow{\Omega}} \neq [\![f]\!]_{\overrightarrow{\Omega}}$ (i.e., $(e \leftrightarrow f)$ is not valid in $\overrightarrow{\Omega}$) then $([e] =)[\![e]\!]_{\overrightarrow{\Omega^*}} \neq [\![f]\!]_{\overrightarrow{\Omega^*}}(= [f])$. For example, suppose that expressions e and $k_i(f)$ happen to satisfy $[\![e]\!]_{\overrightarrow{\Omega'}} = K_i([\![f]\!]_{\overrightarrow{\Omega'}})$ in a particular representation $\overrightarrow{\Omega'}$ (i.e., in a particular context). If there is another knowledge space $\overrightarrow{\Omega}$ which distinguishes these two expressions in the sense that $[\![e]\!]_{\overrightarrow{\Omega}} \neq K_i([\![f]\!]_{\overrightarrow{\Omega}})$, it follows in the universal knowledge space $\overrightarrow{\Omega^*}$ that $[\![e]\!]_{\overrightarrow{\Omega^*}} \neq K_i^*([\![f]\!]_{\overrightarrow{\Omega^*}})$. Put differently, if two expressions satisfy $[\![e]\!]_{\overrightarrow{\Omega^*}} = K_i^*([\![f]\!]_{\overrightarrow{\Omega^*}})$, then it is always the case in any knowledge space that $[\![e]\!]_{\overrightarrow{\Omega}} = K_i([\![f]\!]_{\overrightarrow{\Omega}})$.

Generally, Proposition 2 means that such semantic notions as satisfiability, semantic consequence, and validness in $\overrightarrow{\Omega^*}$ are informationally robust in the sense that they do not depend on any particular representation $\overrightarrow{\Omega}$.

In relation to Theorem 1 and Proposition 2, the following sheds light on how the knowledge space $\overrightarrow{\Omega^*}$ exhausts nature and players' knowledge.

Proposition 3 (Universal Space Exhausts Interactive Knowledge)**.** *We let*

$$\Omega^{**} := \{(s, (\Psi_i)_{i\in I}) \in S \times \mathcal{P}(\mathcal{L})^I \mid \textit{there are a knowledge space } \overrightarrow{\Omega} \textit{ and } \omega \in \Omega \\ \textit{such that } (s, (\Psi_i)_{i\in I}) = (\Theta(\omega), (\{e \in \mathcal{L} \mid \omega \in K_i([\![e]\!]_{\overrightarrow{\Omega}})\})_{i\in I})\}.$$

Then, the following mapping is a bijection:

$$\Omega^* \ni \omega^* \mapsto (\Theta^*(\omega^*), (\{e \in \mathcal{L} \mid \omega^* \in K_i^*([e])\})_{i\in I}) \in \Omega^{**}.$$

The "injection" part of Proposition 3 states that each state ω^* is in a one-to-one relation to a profile of the corresponding nature and each player's knowledge at ω^*. The "surjection" part simply follows from Lemmas 3 and 5.

To conclude this subsection, we pose the following two questions regarding the notion of common knowledge in the universal knowledge space. While the notion of common knowledge is formally defined in Section 4.2, we remark that the common knowledge operator $C : \mathcal{D} \to \mathcal{D}$ is defined in each knowledge space $\overrightarrow{\Omega}$. Thus, for each event $E \in \mathcal{D}$, the set $C(E)$ is the event that E is common knowledge among the players I. In the following, we assume that assumptions on players' knowledge include Monotonicity and Positive Introspection and that they are homogeneously given across players.

First, we remark that the universal knowledge space preserves the common knowledge among players. Proposition 10 in Section 4.2 establishes that, for any knowledge space $\overrightarrow{\Omega}$ (with the above assumptions), an event $D^{-1}([e]) \in \mathcal{D}$ is common knowledge at state $\omega \in \Omega$ iff $[e]$ is common knowledge at $D(\omega) \in \Omega^*$.

Second, at the interpretational level, an implicit assumption often made in state space models of knowledge is the "meta-knowledge" of the model itself (or the primitives of the model themselves).[35] Proposition 13 in Section 5.3 provides one test according to which we (i.e., the outside analysts) can say that the players commonly know the structure of the universal knowledge space. We defer the discussion because we need to formalize the knowledge and common knowledge of the "structure of a model of knowledge."

3.2 Characterizing the Universal Knowledge Space by Coherent Sets of Expressions

In the last subsection, we have established the existence of a universal knowledge space by collecting all possible states that can realize in some knowledge space. Each state in the universal knowledge space consists of a set of expressions (together with a state of nature) satisfiable at some state of some knowledge space. Thus, for a

[35] See, for instance, Aumann [3, 4, 5], Bacharach [6, 7], Binmore and Brandenburger [11], Brandenburger and Dekel [16], BDG [17], Dekel and Gul [20], FGHV [26], Gilboa [32], Pires [70], Roy and Pacuit [71], Tan and Werlang [82].

general regular infinite cardinal κ (especially with $\kappa \geq \aleph_1$ where infinitary operations are allowed), states in the universal κ-knowledge space would generally be different from the collection of "maximally consistent" sets of expressions (together with a state of nature) in some syntax system (see also Moss and Viglizzo [65, 66]).[36]

Now, the question arises as to how (or whether) we can characterize each state ω^* and the set $\overrightarrow{\Omega^*}$ in a somewhat more explicit way. We aim to characterize the universal (κ-)knowledge space Ω^* in terms of the largest collection comprising of "coherent" and "complete" set of expressions.

To that end, recall that each state $\omega^* \in \Omega^*$ contains expressions that hold at ω^* (i.e., $(1, e) \in \omega^*$ iff $\omega^* \in [e]$) as well as the corresponding state of nature $s = \Theta^*(\omega^*)$. This fact suggests the following three ideas as to which expressions a given state ω^* contains. First, every state $\omega^* \in \Omega^*$ contains expressions that hold in any knowledge space (of the given class). That is, each state ω^* contains valid expressions. Second, for any expression e, if a state ω^* contains e then it does not contain $(\neg e)$. In other words, each ω^* is *coherent*. Third, on the other hand, if a state ω^* does not contain e then it contains $(\neg e)$. That is, each ω^* is *complete*. The terminologies of coherence and completeness in this context (i.e., coherence and completeness of each state in terms of the expressions $\mathcal{L}$) are from Aumann [5].

Here, we formally link the universal knowledge space obtained in the previous section with Aumann's [5] construction of what he calls a canonical knowledge system (of a finitary epistemic $S5$ logic) by generalizing Aumann's [5] idea in the following sense. We show that the universal knowledge space Ω^* is written as the largest set with the following two properties: (i) its element (state) is a complete and coherent set of expressions together with the corresponding state of nature; and (ii) its element (state) reflects the given properties of knowledge. Thus, we establish an alternative characterization of the universal knowledge space obtained in Section 3.1. This characterization holds irrespective of a given cardinality κ and assumptions on players' knowledge.

Theorem 2 (Largest Set of Complete and Coherent Expressions)**.** *The set Ω^* obtained in Section 3.1 is the largest set satisfying the following: (i) Ω^* satisfies all the conditions specified below; and (ii) for any set Ω satisfying all the conditions below, there is a knowledge space $\overrightarrow{\Omega}$ such that its description map $D_{\overrightarrow{\Omega}} : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ is an inclusion map (and thus $\Omega \subseteq \Omega^*$).*

1. *Each element $\omega \in \Omega$ is a subset of $S \sqcup \mathcal{L}$ with the following properties.*
   (a) *There is a unique $s \in S$ such that $(0, s) \in \omega$. Moreover, for such an $s \in S$, $(1, E) \in \omega$ for all $E \in \mathcal{A}_S$ with $s \in E$.*[37]

[36] Indeed, the discrepancy between a semantic notion of satisfiability and a syntactic notion of maximal consistency would emerge even at the level of a propositional logic when infinitary operations are allowed (Karp [45]). See also Heifetz [37] and Meier [57] for this point.

[37] By Condition (1e) below, Condition (1a) implies that, for a unique $s \in S$ with $(0, s) \in \omega$, we have $(1, E) \in \omega$ iff $s \in E$ for any $E \in \mathcal{A}_S$. Especially, we have $(1, S) \in \omega$ and $(1, \emptyset) \notin \omega$.

(b) *Depending on the notion of knowledge, ω contains any instance of the following expressions.*

   i. *No-Contradiction Axiom:* $(\emptyset \leftrightarrow k_i(\emptyset))$.

   ii. *Consistency:* $(k_i(e) \rightarrow (\neg k_i)(\neg e))$.

   iii. *Non-empty λ-Conjunction:* $((\bigwedge_{e\in\mathcal{E}} k_i(e)) \rightarrow k_i(\bigwedge \mathcal{E}))$ *with* $0 < |\mathcal{E}| < \lambda(\leq \kappa)$.

   iv. *Necessitation:* $k_i(S)$.

   v. *Truth Axiom:* $(k_i(e) \rightarrow e)$.

   vi. *Positive Introspection:* $(k_i(e) \rightarrow k_i k_i(e))$.

   vii. *Negative Introspection:* $((\neg k_i)(e) \rightarrow k_i(\neg k_i)(e))$.

(c) *If* $(1, e) \in \omega$ *and* $(1, (e \rightarrow f)) \in \omega$ *then* $(1, f) \in \omega$.

(d) *For any* $\mathcal{E}$ *with* $\mathcal{E} \subseteq \mathcal{L}$ *and* $|\mathcal{E}| < \kappa$, $(1, \bigwedge \mathcal{E}) \in \omega$ *iff* $(1, e) \in \omega$ *for all* $e \in \mathcal{E}$.

(e) *Coherency: For each* $e \in \mathcal{L}$, *if* $(1, (\neg e)) \in \omega$ *then* $(1, e) \notin \omega$.

(f) *Completeness: For each* $e \in \mathcal{L}$, *if* $(1, e) \notin \omega$ *then* $(1, (\neg e)) \in \omega$.

2. Ω *satisfies the following conditions.*

   (a) *If expressions* e *and* f *are such that* $(1, (e \leftrightarrow f)) \in \omega$ *for all* $\omega \in \Omega$, *then* $(1, (k_i(e) \leftrightarrow k_i(f))) \in \omega$ *for all* $\omega \in \Omega$.

   (b) *Suppose that Monotonicity is imposed on i's knowledge. If expressions* e *and* f *are such that* $(1, (e \rightarrow f)) \in \omega$ *for all* $\omega \in \Omega$, *then* $(1, (k_i(e) \rightarrow k_i(f))) \in \omega$ *for all* $\omega \in \Omega$.

   (c) *Suppose that the Kripke property is imposed on i's knowledge. Then,* $(1, k_i(e)) \in \omega$ *for any* $e \in \mathcal{L}$ *and* $\omega \in \Omega$ *with the following condition: if* $\omega' \in \Omega$ *satisfies* $(1, f) \in \omega'$ *for all* $f \in \mathcal{L}$ *with* $(1, k_i(f)) \in \omega$ *then* $(1, e) \in \omega'$.

In Theorem 2, Condition (1a) states that each state of the world ω describes a corresponding state of nature s in a well-defined manner. Also, the state of the world ω contains those events of nature $E \in \mathcal{A}_S$ that are true at s (i.e., $s \in E$). Conditions (1c) through (1f) are logical requirements on how the world is described.

Condition (2a) requires that if two expressions e and f are equivalent in the sense that $(e \leftrightarrow f)$ is in any state of the world then expressions $k_i(e)$ and $k_i(f)$ are equivalent in the same sense. This condition allows us to define players' knowledge operators in a way such that if two expressions e and f correspond to the same event then the events associated with the knowledge of e and f are the same. We have seen in Lemma 3 that $\overrightarrow{\Omega}^*$ satisfies this property.

Each condition in (1b) describes how each state of the world describes the corresponding property of players' knowledge. We have seen, in Proposition 1, a related characterization for Consistency and Negative Introspection. Monotonicity and the

Kripke property are described, in (2b), and (2c), by conditions on the entire states of the world Ω. If these conditions are satisfied in Ω, then it means that assumptions on players' knowledge are encoded within Ω itself in the sense that we can induce players' knowledge operators which satisfy the given assumptions.

3.3 Comparison with the Previous Negative Results

We discuss how domain specifications together with an appropriate notion of knowledge morphism (which reflects domain specifications) have an essential role in establishing the existence of a universal knowledge space. The basic idea is that, for any given infinite regular cardinal κ, any κ-knowledge space can capture players' interactive knowledge of the ordinal depth up to κ. Theorem 1 establishes the existence of a universal κ-knowledge space within such class of κ-knowledge spaces.

To compare our existence result with the previous negative results, we look at the notion of a rank of a standard partitional knowledge space (HS [38]).[38] HS [38] demonstrate that there is no universal standard partitional knowledge space on the following two grounds. First, a knowledge morphism preserves the ranks. Second, there is a standard partitional knowledge space with arbitrary high rank. These facts imply that, for any candidate universal standard partitional knowledge space, there exists a standard partitional knowledge space which has a higher rank and thus the candidate space must not be universal.

We extend their notion of a rank to that of a κ-rank of a κ-knowledge space, because not all subsets are expressible within the κ-complete algebra of a given κ-knowledge space. We see (i) that a knowledge morphism preserves the κ-ranks but (ii) that the κ-rank of any κ-knowledge space is at most κ. Below is the definition of the κ-rank of a κ-knowledge space.

Definition 10 (κ-Rank: Maximal Ordinality of Interactive Knowledge in Each Knowledge Space)**.** *Let $\overrightarrow{\Omega} := \langle (\Omega, \mathcal{D}), (K_i)_{i \in I}, \Theta \rangle$ be a κ-knowledge space of I on $(S, \mathcal{A}_S)$. The κ-rank of $\overrightarrow{\Omega}$ is defined as the least ordinal α such that $\mathcal{C}_\alpha = \mathcal{C}_{\alpha+1}$, where the sequence $(\mathcal{C}_\alpha)_\alpha$ is defined as follows:*

$$\mathcal{C}_\alpha := \begin{cases} \mathcal{A}_\kappa(\{\Theta^{-1}(E) \in \mathcal{D} \mid E \in \mathcal{A}_S\}) (= \{\Theta^{-1}(E) \in \mathcal{D} \mid E \in \mathcal{A}_S\}) & \text{if } \alpha = 0 \\ \mathcal{A}_\kappa\left(\left(\bigcup_{\beta < \alpha} \mathcal{C}_\beta \right) \cup \bigcup_{i \in I} \{K_i(E) \in \mathcal{D} \mid E \in \bigcup_{\beta < \alpha} \mathcal{C}_\beta\} \right) & \text{if } \alpha > 0 \end{cases}.$$

With this definition in mind, we establish the following.

Proposition 4 (κ-Rank of a Universal κ-Knowledge Space)**.** *1. If $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a knowledge morphism between κ-knowledge spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$, then the κ-rank of $\overrightarrow{\Omega'}$ is at least as high as that of $\overrightarrow{\Omega}$.*

[38] Fagin [25] considers a closely related concept, "distinguishing ordinals," in his modal logic framework.

2. *The κ-rank of any κ-knowledge space $\overrightarrow{\Omega}$ is at most κ.*

The first part of Proposition 4 states that a knowledge morphism between κ-knowledge spaces preserves the κ-ranks. The second part, on the other hand, asserts the difference between κ-knowledge spaces and standard knowledge spaces whose domains are power sets. Namely, the κ-rank of any κ-knowledge space is at most κ. This result hinges on the fact that the set of expressions $\mathcal{L}(= \mathcal{L}^I_\kappa(\mathcal{A}_S))$ consists of expressions that involve player's interactive knowledge of the ordinality up to κ (Remark 1).[39]

Specifically, we define $\mathcal{D}_\alpha = \{[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid e \in \mathcal{L}_\alpha\}$ for each ordinal $\alpha \leq \kappa$, where $\mathcal{L}_\alpha$ is defined as in Remark 1. Then, we show in the proof that $\mathcal{D}_\alpha = \mathcal{C}_\alpha$ for each ordinal $\alpha \leq \kappa$. Then, it follows that $\mathcal{D}_\kappa = \mathcal{C}_\kappa = \mathcal{C}_{\kappa+1}$, i.e., the κ-rank of $\overrightarrow{\Omega}$ is at most κ. We also remark that the κ-complete sub-algebra $\{[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid e \in \mathcal{L}\}$ is equal to $\mathcal{C}_\kappa$. That is, the notion of non-redundancy can be examined through $\mathcal{C}_\kappa$.

We make two remarks regarding the second part of Proposition 4. First, for an infinite regular cardinal κ, HS's [38] non-existence argument does not apply to a given class of κ-knowledge spaces. Second, it is important to take care of all the κ levels of interactive knowledge in order to incorporate possible "discontinuity" of knowledge.[40]

By fixing the language that the players are allowed to use in reasoning about their interactive knowledge (within the ordinality of κ), a knowledge morphism (a description map) preserves interactive knowledge in a given (κ-)knowledge space to the universal (κ-)knowledge space. At the same time, such preservation concerns only to the extent that (κ-)expressions are preserved.

Finally, we discuss how our results reconcile with the existence of a universal partitional κ-knowledge space. First, fix any infinite regular cardinal κ. Since we can identify the conditions on each player's knowledge operator under which her knowledge is induced from a partition (namely, Truth Axiom, Negative Introspection, and the Kripke property), Theorem 1 demonstrates that there is a universal partitional κ-knowledge space. In partitional κ-knowledge spaces, the conjunction property is generally restricted to κ-Conjunction.

On the contrary, consider the ∞-knowledge spaces of I on $(S, \mathcal{P}(S))$ (with $|S| \geq 2$ and $|I| \geq 2$).[41] In this case, the notion of an ∞-rank is equivalent to that defined by HS [38]. Thus, contrary to the case where κ is an infinite regular cardinal, there is no universal ∞-knowledge space on $(S, \mathcal{P}(S))$ satisfying all the logical and introspective properties (provided that $|S| \geq 2$ and $|I| \geq 2$). Moreover, this non-existence result

[39] In the hierarchical construction of a universal knowledge space in Section 6, we also define the space consisting of hierarchies of knowledge up to the ordinality of κ.

[40] HS [38, 39] attribute the non-existence of a universal (standard partitional) knowledge space to the "lack of continuity" of knowledge. On a related point, FHV [28], Fagin [25], FGHV [26], and HS [41] attribute the non-existence of the space of all coherent hierarchies of knowledge to the lack of "continuity" property of knowledge structures, as opposed to that of σ-additive probability measures. See also Meier [55, 56] for the discussion of the use of infinitary expressions.

[41] Note that if $\mathcal{A}_S$ satisfies the separative property (see Footnote 24) then $\mathcal{A}_S = \mathcal{P}(S)$.

shows that there is no universal ∞-knowledge space in a category of knowledge spaces on $(S, \mathcal{P}(S))$ which contains, as a subclass, the category of knowledge spaces satisfying all the logical and introspective properties. With respect to our previous discussion, the collection of "∞-expressions" (Definition 6 with $\kappa = \infty$) would be too large to be a set (in the realm of the standard set theory).

4 Knowledge and Common Knowledge in Knowledge Spaces

We study how we can capture players' knowledge and common knowledge in each knowledge space. First, in Section 4.1, we represent each player's knowledge by a set algebra if her knowledge satisfies Truth Axiom, Monotonicity, and Positive Introspection. The usefulness of this representation would lie in the fact that if each player's knowledge also satisfies Negative Introspection then her knowledge is represented as a sub-algebra without imposing an assumption on the cardinality of the sub-algebra.

Second, in Section 4.2, we introduce the notion of common knowledge regardless of assumptions on players' knowledge, where note that the discussion of the common knowledge pre-supposes notations in Section 4.1. The notion of common knowledge is used to ask the sense in which the players commonly know the structure of a universal knowledge space (Proposition 13 in Section 5.3).

This section aims to have representations of players' knowledge and common knowledge in the framework of knowledge spaces. In this regard, we generalize, in Appendix A.7.1, possibility correspondence models of knowledge under Monotonicity in order to claim that our framework of knowledge spaces admits a variety of knowledge representations. We also connect our framework to the literature of knowledge representation in psychology (Doignon and Falmagne [22, 23]) in Appendix A.7.4.

Note that, in this paper, we represent players' knowledge by (i) knowledge operators (Section 2.2), (ii) set algebras (Section 4.1), (iii) knowledge-type mappings (which are qualitative analogues of belief-type mappings, in Section 5.1), and generalized possibility correspondences (Appendix A.7.1). Under appropriate conditions, one framework is shown to be equivalent to another. This means that a universal knowledge space exists under various frameworks (with various combinations of properties on knowledge).

4.1 Set-algebraic Representation of Knowledge

Here, we provide a set-algebraic representation of knowledge. This embodies the intuition that the content of knowledge can be captured by a sub-collection of a given domain. To that end, we introduce the following two definitions. First, we say that an event E is *self-evident* to player i if $E \subseteq K_i(E)$.[42] In words, E is self-evident to i if she knows E whenever E obtains. We denote the collection of events which are self-evident to i, which we call i's *self-evident collection*, by $\mathcal{J}_{K_i} := \{E \in \mathcal{D} \mid E \subseteq K_i(E)\}$.[43]

Second, we say that a given sub-collection $\mathcal{J}_i$ of $\mathcal{D}$ satisfies the *maximality property* (w.r.t. $\mathcal{D}$) if $\emptyset \in \mathcal{J}_i$ and for any $E \in \mathcal{D}$, the largest element of $\mathcal{J}_i$ contained in E, $\max\{F \in \mathcal{D} \mid F \in \mathcal{J}_i \text{ and } F \subseteq E\}$, exists in $\mathcal{J}_i$.[44]

With these two definitions in mind, we establish the following two results. First, we show that properties of knowledge operators are equivalently expressed as set-algebraic properties of self-evident collections when Truth Axiom, Positive Introspection, and Monotonicity are imposed.

Specifically, given a knowledge operator K_i which satisfies Truth Axiom, Positive Introspection, and Monotonicity, the event $K_i(E)$ is the largest (in the sense of set inclusion) self-evident event to i which is contained in E. Other properties of K_i are translated into set-algebraic properties of $\mathcal{J}_{K_i}$. For example, Negative Introspection is translated as the closure under complementation. The self-evident collection $\mathcal{J}_{K_i}$, in turn, induces the knowledge operator through $K_i = K_{\mathcal{J}_{K_i}}(E) := \max\{F \in \mathcal{J}_{K_i} \mid F \subseteq E\}$.

Our second result is that, for any given sub-collection $\mathcal{D}'$ of $\mathcal{D}$, we can express the smallest sub-collection $\mathcal{J}(\mathcal{D}')$ containing $\mathcal{D}'$ and satisfying the maximality property. Thus, we can induce an informational content to a given sub-collection $\mathcal{D}'$ of events. Note that we will use this concept to define the notion of common knowledge in the next subsection.

Now, we go on to our first main result: a knowledge space admits representations of players' knowledge in terms of self-evident collections.

Proposition 5 (Equivalence between Knowledge Operators and Self-Evident Collections)**.** *Fix a κ-complete algebra $(\Omega, \mathcal{D})$.*

1. *For a given knowledge operator $K_i : \mathcal{D} \to \mathcal{D}$, the self-evident collection $\mathcal{J}_{K_i}$ satisfies the following.*

[42] Binmore and Brandenburger [11] call such an event to be a truism.

[43] We have a technical remark. If we express each player's knowledge by a self-evident collection as opposed to a knowledge operator, we specify it as a sub-collection of a given domain. Thus, two collections $\mathcal{J}_{K_i}$ on $\mathcal{D}$ and $\mathcal{J}'_{K'_i}$ on $\mathcal{D}'$ are considered to be different as long as the domains $\mathcal{D}$ and $\mathcal{D}'$ differ, even if $\mathcal{J}_{K_i}$ and $\mathcal{J}'_{K'_i}$ are extensionally equivalent (have the same elements).

[44] The terminology is due to Samet [78], where the maximality property is defined for a collection of events which forms an ($\aleph_0$-complete) sub-algebra of an ($\aleph_0$-complete) algebra $\mathcal{D}$. As Proposition 5 demonstrates, this corresponds to the knowledge (defined on an algebra) which satisfies Truth Axiom, Monotonicity, (Positive Introspection, Finite Conjunction), and Negative Introspection. In the sense that Negative Introspection is not necessarily imposed, the maximality property of a self-evident collection is generally not identical with the notion of "relative completeness" of a Boolean sub-algebra (Halmos [33, Section 4]).

(a) If K_i satisfies Truth Axiom, Monotonicity, and Positive Introspection, then $\mathcal{J}_{K_i}$ satisfies the maximality property in the following sense. For any $E \in \mathcal{D}$,

$$K_i(E) = \max\{F \in \mathcal{D} \mid F \in \mathcal{J}_{K_i} \text{ and } F \subseteq E\} \qquad (3)$$
$$(= \{\omega \in \Omega \mid \text{there is } F \in \mathcal{J}_{K_i} \text{ with } \omega \in F \text{ and } F \subseteq E\}).$$

(b) If K_i satisfies Non-empty λ-Conjunction (with $\lambda \leq \kappa$), then $\mathcal{J}_{K_i}$ is closed under non-empty λ-intersection.

(c) If K_i satisfies Necessitation, then $\Omega \in \mathcal{J}_{K_i}$.

(d) If K_i satisfies Truth Axiom and Negative Introspection, then $\mathcal{J}_{K_i}$ is closed under complementation.

2. Conversely, for a given sub-collection $\mathcal{J}_i$ of $\mathcal{D}$ which satisfies the maximality property, the operator $K_{\mathcal{J}_i} : \mathcal{D} \to \mathcal{D}$ satisfies the converse of the above assertion in the following sense, where $K_{\mathcal{J}_i}(E) := \max\{F \in \mathcal{J}_i \mid F \subseteq E\}$.

(a) $K_{\mathcal{J}_i}$ satisfies Truth Axiom, Monotonicity, and Positive Introspection.

(b) If $\mathcal{J}_i$ is closed under non-empty λ-intersection (with $\lambda \leq \kappa$), then $K_{\mathcal{J}_i}$ satisfies Non-empty λ-Conjunction.

(c) If $\Omega \in \mathcal{J}_i$ then $K_{\mathcal{J}_i}$ satisfies Necessitation.

(d) If $\mathcal{J}_i$ is closed under complementation then $K_{\mathcal{J}_i}$ satisfies Negative Introspection.

3. Starting with K_i, we have $K_{\mathcal{J}_{K_i}} = K_i$. Likewise, $\mathcal{J}_i$ induces $\mathcal{J}_i = \mathcal{J}_{K_{\mathcal{J}_i}}$.

4. If $\kappa = \infty$, then $\mathcal{J}_{K_i}$ ($\mathcal{J}_i$) satisfies the maximality property iff $\mathcal{J}_{K_i}$ ($\mathcal{J}_i$) is closed under arbitrary union. Generally, if $\mathcal{J}_{K_i}$ ($\mathcal{J}_i$) satisfies the maximality property, then $\mathcal{J}_{K_i}$ ($\mathcal{J}_i$) is closed under κ-union.

An immediate corollary of Proposition 5 is that Negative Introspection, together with Truth Axiom and Monotonicity, imply all the other logical and introspective properties of knowledge postulated in Definition 2.

Corollary 2 (Negavtive Introspection Implies Conjunction)**.** *Let $\overrightarrow{\Omega}$ be a κ-knowledge space such that $K_i : \mathcal{D} \to \mathcal{D}$ satisfies Truth Axiom, Monotonicity, and Negative Introspection. Then, K_i also satisfies κ-Conjunction including Necessitation (as well as No-Contradiction Axiom, Consistency, and Positive Introspection).*

We make six additional remarks regarding Proposition 5. First, the maximality property ensures that $K_i(E) = \bigcup\{F \in \mathcal{D} \mid F \in \mathcal{J}_{K_i} \text{ and } F \subseteq E\} \in \mathcal{D}$ for each $E \in \mathcal{D}$, in spite of the fact that the self-evident collection (as well as the domain $\mathcal{D}$) may not necessarily be closed under arbitrary union.

Second, the self-evident collection satisfies $\mathcal{J}_{K_i} = \{K_i(E) \in \mathcal{D} \mid E \in \mathcal{D}\}$ as long as K_i satisfies Truth Axiom and Positive Introspection. Thus, it comprises solely of the largest self-evident event $K_i(E)$ contained in E for each $E \in \mathcal{D}$.

Third, all of Truth Axiom, Positive Introspection, and Monotonicity are essential in establishing the equivalence between a knowledge operator and a self-evident collection. If any one condition is absent, there is a simple example where $K_i \neq K_i'$ but $\mathcal{J}_{K_i} = \mathcal{J}_{K_i'}$. See Remark A.4 in the Appendix for concrete examples.

Fourth, Proposition 5 provides the sense in which knowledge takes a form of various set-algebras. For example, a self-evident collection in an ∞-knowledge space becomes a (sub-)topology ((sub-)topology closed under arbitrary intersection) under Truth Axiom, Monotonicity, Positive Introspection, and Finite (Arbitrary) Conjunction including Necessitation.[45]

Moreover, if knowledge satisfies Truth Axiom, Monotonicity, and Negative Introspection (again, recall Corollary 2) in a κ-knowledge space, then the self-evident collection becomes a κ-complete sub-algebra. To restate, for $\kappa \in \{\aleph_0, \aleph_1, \infty\}$, knowledge takes a form of a sub-algebra, σ-sub-algebra, and complete sub-algebra, respectively. Thus, a self-evident collection as a σ-sub-algebra could induce the conditional probability (belief). Since such σ-sub-algebra satisfies the maximality property, we would not need to impose an extra assumption on its cardinality.

At the same time, if a domain of knowledge is a complete algebra in the presence of probabilistic beliefs, then we would have to be careful about σ-measurability. This is because, knowledge, which takes a form of a complete sub-algebra, is usually richer than a σ-algebra.[46]

Fifth, a self-evident collection may not necessarily be closed under complementation in the absence of Negative Introspection, and hence we need to take care of this asymmetry when we interpret self-evident events. To see this point, observe that the possibility operator L_{K_i} satisfies $E^c \in \mathcal{J}_{K_i}$ iff $L_{K_i}(E) \subseteq E$ (i.e., E is true whenever i considers it possible). Put differently, when $E \in \mathcal{J}_{K_i}$ but $E^c \notin \mathcal{J}_{K_i}$, it is not the case that player i knows E is false (E^c is true) whenever E is false at ω. Colloquially, she

[45] If an i's knowledge satisfies Truth Axiom, Monotonicity, Positive Introspection, and Arbitrary Conjunction (including Necessitation) in an ∞-knowledge space, then it will turn out (an an implication of Proposition 12 in Section 5.2) that $b_{K_i} : \Omega \to \mathcal{D}$ is a reflexive and transitive possibility correspondence. See Footnote 3 for the literature on non-partitional possibility correspondence models of knowledge.

[46] That is, a complete algebra may be "too large" for some σ-additive measure to be well defined. Also, assuming the Kripke property, an i's information set $b_{K_i}(\omega)$ may be null so that we have to take care of Bayesian updating on such a null set if i's posterior belief is derived from a prior distribution (see, for example, Brandenburger and Dekel [15] for the use of regular conditional probability measures or Nielsen [68] for enriching the underlying state space). On the other hand, modeling information by a σ-algebra may be problematic due to the fact that σ-algebras generated by partitions are not necessarily monotonic in the fineness of partitions (see Dubra and Echenique [24], Hérves-Beloso and Monteiro [42], Nielsen [68], Stinchcombe [81] and the references therein). We will shortly discuss how the maximality property can capture the preservation of informational contents.

cannot conclude that E is false just by the fact that she does not observe E.[47]

If we start with the possibility operator, then the corresponding dual collection $\overline{\mathcal{J}}_{K_i} := \{F \in \mathcal{D} \mid F^c \in \mathcal{J}_{K_i}\}$ satisfies the minimality property in the following sense:

$$L_{K_i}(E) = \min\{F \in \overline{\mathcal{J}}_{K_i} \mid E \subseteq F\}(\in \overline{\mathcal{J}}_{K_i}).$$

That is, for any $E \in \mathcal{D}$, the dual collection $\overline{\mathcal{J}}_{K_i}$ always has the minimum event containing E. Since $\mathcal{J}_{K_i}$ is not necessarily closed under complementation, however, $\mathcal{J}_{K_i}$ does not necessarily satisfy the minimality property in the form of $L_{K_i}(E) = \min\{F \in \mathcal{J}_{K_i} \mid E \subseteq F\}$.[48] Without Negative Introspection of K_i, these two minimality properties do not necessarily coincide with each other.

Sixth, set-inclusion between self-evident collections naturally gives (global) "knowledgeability" relation. Namely, suppose that knowledge operators K_i and K_j satisfy Truth Axiom, Positive Introspection, and Monotonicity (equivalently, self-evident collections $\mathcal{J}_i$ and $\mathcal{J}_j$ satisfy the maximality property). Then, we have

$$\mathcal{J}_i \subseteq \mathcal{J}_j \textit{ iff } K_i(\cdot) \subseteq K_j(\cdot).$$

Generally, the second condition implies the first without any assumption on knowledge operators. This follows because $E \subseteq K_i(E) \subseteq K_j(E)$ for any $E \in \mathcal{J}_{K_i}$. If the self-evident collections represent players' knowledge (through the maximality property), then the first condition implies the second.

Now, we go on to our second result. For any given sub-collection $\mathcal{D}'$ of $\mathcal{D}$, we consider the smallest collection containing $\mathcal{D}'$, satisfying the maximality property and possibly various other logical and introspective properties. This observation enables us to give an informational content to any given sub-collection $\mathcal{D}'$, for example, in the following cases. First, for a given σ-algebra, we can consider the smallest σ-algebra containing the original one and satisfying the maximality property. Second, if a player's knowledge is given by her self-evident collection and she "learns" a certain set of events, then we can obtain the new smallest self-evident collection containing both collections. Third, for a given set of players, we can get the smallest self-evident collections containing the intersection of self-evident collections (i.e., the "publicly

[47] On the other hand, for any $E \in \mathcal{J}_{K_i}$ with $E^c \in \mathcal{J}_{K_i}$, player i knows whether E is true or not at any state (i.e., $\Omega = K_i(E) \cup K_i(E^c)$). In other words, if $\mathcal{J}_{K_i}$ forms a κ-complete sub-algebra of $\mathcal{D}$, then $\mathcal{J}_{K_i}$ comprises of an event E such that i knows whether E is true or not at any state. This formalizes the (common) sense in which knowledge is interpreted as a sub-algebra of a given domain (see, for example, Hérves-Beloso and Monteiro [42] and Stinchcombe [81]). Indeed, a self-evident event conicides with an event which Hérves-Beloso and Monteiro [42] call an informed set (with respect to a partition) in the following sense. If an agent's knowledge is represented by a partition (i.e., her knowledge satisfies the Kripke property, Truth Axiom, and Negative Introspection), then an event is self-evident to her iff it is an informed set with respect to her information partition (where technically we allow each partition cell $b_{K_i}(\omega)$ not to be an event).

[48] As an example, consider $\Omega = \{\omega_1, \omega_2\}$, $\mathcal{D} = \mathcal{P}(\Omega)$, and $\mathcal{J}_{K_i} = \{\emptyset, \{\omega_1\}\}$. We have $L_{K_i}(\{\omega_1\}) = \min\{F \in \overline{\mathcal{J}}_{K_i} \mid \{\omega_1\} \subseteq F\} = \Omega \neq \{\omega_1\} = \min\{F \in \mathcal{J}_{K_i} \mid E \subseteq F\}$. This happens because $\{\omega_1\}^c$ is not self-evident.

evident" (Milgrom [60]) collection $\bigcap_{i\in I}\mathcal{J}_i$). This is the "infimum" informational content associated with the group of players. We will, in fact, formalize the notion of common knowledge in Section 4.2 by slightly amending this observation. Fourth, we can also pool players' knowledge $\bigcup_{i\in I}\mathcal{J}_i$ and consider the smallest self-evident collection containing the pooling of thier knowledge. This corresponds to the notion of distributed knowledge (Halpern and Moses [34]).

Proposition 6 (Smallest Collection Satisfying Maximality). *Let $\mathcal{D}'$ be a sub-collection of a κ-complete algebra $\mathcal{D}$ on Ω.*

1. *The smallest collection containing $\mathcal{D}'$ and satisfying the maximality property is*

$$\mathcal{J}(\mathcal{D}') := \{E \in \mathcal{D} \mid \textit{if } \omega \in E \textit{ then there is } F \in \mathcal{D}' \textit{ with } \omega \in F \subseteq E\}. \quad (4)$$

2. *The smallest collection containing $\mathcal{D}' \cup \{\Omega\}$ and satisfying the maximality property is $\mathcal{J}_{Nec}(\mathcal{D}') := \mathcal{J}(\mathcal{D}' \cup \{\Omega\}) = \mathcal{J}(\mathcal{D}') \cup \{\Omega\}$.*

3. *Fix $\lambda \leq \kappa$. If $\mathcal{D}'$ is closed under non-empty λ-intersection, so does $\mathcal{J}(\mathcal{D}')$. Generally, the smallest collection containing $\mathcal{D}'$, satisfying the maximality property, and closed under non-empty λ-intersection is*

$$\mathcal{J}_{\lambda\text{-}Con}(\mathcal{D}') := \mathcal{J}(\{E \in \mathcal{D} \mid E = \bigcap \mathcal{E} \textit{ for some } \mathcal{E} \subseteq \mathcal{D}' \textit{ with } 0 < |\mathcal{E}| < \lambda\})$$
$$= \{E \in \mathcal{D} \mid \textit{if } \omega \in E \textit{ then there is } \mathcal{E} \subseteq \mathcal{D}' \textit{ with } 0 < |\mathcal{E}| < \lambda \textit{ and } \omega \in \bigcap \mathcal{E} \subseteq E\}.$$

   Likewise, the smallest collection containing $\mathcal{D}'$, satisfying the maximality property and closed under λ-intersection is

$$\mathcal{J}_{\lambda\text{-}Con,\ Nec}(\mathcal{D}') := \mathcal{J}(\{E \in \mathcal{D} \mid E = \bigcap \mathcal{E} \textit{ for some } \mathcal{E} \subseteq \mathcal{D}' \textit{ with } |\mathcal{E}| < \lambda\})$$
$$= \{E \in \mathcal{D} \mid \textit{if } \omega \in E \textit{ then there is } \mathcal{E} \subseteq \mathcal{D}' \textit{ with } |\mathcal{E}| < \lambda \textit{ and } \omega \in \bigcap \mathcal{E} \subseteq E\}.$$

4. *The collection $\mathcal{J}(\mathcal{D}')$ does not necessarily inherit the closure under complementation from $\mathcal{D}'$. The smallest collection $\mathcal{J}_{NI}(\mathcal{D}')$ containing $\mathcal{D}'$, satisfying the maximality property, and closed under complementation is given as follows.*[49] *We start with defining auxiliary collections. Let $\mathcal{C}_0 = \mathcal{D}'$. For a successor ordinal $\alpha = \beta + 1$, we let $\mathcal{C}_{\beta+1} := \mathcal{J}(\mathcal{C}_\beta) \cup \{E \in \mathcal{D} \mid E^c \in \mathcal{C}_\beta\}$. For a limit ordinal α, we let $\mathcal{C}_\alpha = \bigcup_{\beta<\alpha}\mathcal{C}_\beta$. Observe that $\mathcal{C}_\alpha \subseteq \mathcal{D}$ for all α. Letting α be the least ordinal with $\mathcal{C}_\alpha = \mathcal{C}_{\alpha+1}$, we define $\mathcal{J}_{NI}(\mathcal{D}') := \mathcal{C}_\alpha$.*

5. *Suppose that $\kappa = \infty$. Then, we have*

$$\mathcal{J}(\mathcal{D}') = \bigcap\{\mathcal{J}' \in \mathcal{P}(\mathcal{D}) \mid \mathcal{D}' \subseteq \mathcal{J}' \textit{ and } \mathcal{J}' \textit{ is closed under arbitrary union}\}, \textit{ and}$$
$$\mathcal{J}_{NI}(\mathcal{D}') = \mathcal{A}_\infty(\mathcal{D}').$$

[49] Note that, by Proposition 5, $\mathcal{J}_{\mathrm{NI}}(\mathcal{D}')$ is also closed under κ-intersection.

We have the following four remarks on Proposition 6. First, we can write the knowledge operator associated with $\mathcal{J}(\mathcal{D}')$ through the maximality property as follows:

$$K_{\mathcal{J}(\mathcal{D}')}(E) = \{\omega \in \Omega \mid \text{there is } F' \in \mathcal{D}' \text{ with } \omega \in F' \subseteq E\} \text{ for each } E \in \mathcal{D}.$$

Second, Proposition 6 also implies that, with a κ-complete algebra $(\Omega, \mathcal{D})$ fixed, the family of all self-evident collections forms a complete lattice with respect to the set inclusion (i.e., the knowledgeability relation). Especially, for a given profile of self-evident collections $(\mathcal{J}_{K_i})_{i \in I}$, the infimum is given by $\mathcal{J}(\bigcap_{i \in I} \mathcal{J}_{K_i})$ while the supremum is given by $\mathcal{J}(\bigcup_{i \in I} \mathcal{J}_{K_i})$.

Third, it can be seen that the dual collection $\overline{\mathcal{J}(\mathcal{D}')}$ (i.e., $E \in \overline{\mathcal{J}(\mathcal{D}')}$ iff $E^c \in \mathcal{J}(\mathcal{D}')$) is given by

$$\overline{\mathcal{J}(\mathcal{D}')} = \{E \in \mathcal{D} \mid \text{if } E \cap F \neq \emptyset \text{ for any } (\omega, F) \in \Omega \times \mathcal{D}' \text{ with } \omega \in F, \text{ then } \omega \in E\}.$$

Fourth, we examine the sense in which $\mathcal{J}$ preserves collection of events. A part of the argument made by Dubra and Echenique [24] that σ-algebras are not necessarily adequate for modeling information is that the operation of taking the smallest σ-algebra does not necessarily preserve set inclusion with respect to partitions. Dubra and Echenique [24, Theorem A] and Hérves-Beloso and Monteiro [42, Propositions 1 and 5] study the operations that preserve an informational content (a partition) and establish Blackwell's theorem connecting the preferences over signals with information partitions. Here, we can see from Equation (4) that $\mathcal{J}$ preserves collections of events (including partitions) in the following sense.

Corollary 3 (Preservation of Information)**.** *Let $(\Omega, \mathcal{D})$ be a κ-complete algebra. Let $\mathcal{E}$ and $\mathcal{E}'$ be such that $\mathcal{E}, \mathcal{E}' \subseteq \mathcal{D}$. The following are equivalent.*

1. *$\mathcal{J}(\mathcal{E}) \subseteq \mathcal{J}(\mathcal{E}')$.*
2. *For any $E \in \mathcal{E}$ and $\omega \in E$, there is $E' \in \mathcal{E}'$ such that $\omega \in E' \subseteq E$.*

To conclude this subsection, we remark that Proposition A.3 in the Appendix characterizes knowledge morphisms in terms of self-evident collections. Thus, under the assumption that each player's knowledge satisfies Truth Axiom, Monotonicity, and Positive Introspection, players' self-evident collections can be given as a primitive of a knowledge space.

4.2 Common Knowledge

Here, we aim to formalize the notion of common knowledge among the group of players I by defining the common knowledge operator $C : \mathcal{D} \to \mathcal{D}$ in any given knowledge space.[50] Throughout this subsection, we simply assume that a given κ-knowledge space $\overrightarrow{\Omega}$ of I on $(S, \mathcal{A}_S)$ satisfies $|I| < \kappa$.

In order to define the notion of common knowledge for any domain and irrespective of assumptions on players' knowledge, we start with defining the following preliminary definition. An event $F \in \mathcal{D}$ is called a *common basis* (among I) if $F \subseteq K_I(E)$ for any $F \in \mathcal{D}$ with $F \subseteq E$, where $K_I(E) := \bigcap_{i \in I} K_i(E)$ is the event that every player in I knows E. Thus, an event F is a common basis if everybody knows any logical implication of F whenever F is true. We denote by $\mathcal{J}_I$ the collection of an event that forms a common basis among I. Clearly, if F is a common basis then F is publicly evident (i.e., $F \subseteq K_I(F)$). That is, $\mathcal{J}_I \subseteq \bigcap_{i \in I} \mathcal{J}_{K_i}$. Conversely, if every player's knowledge satisfies Monotonicity then these two notions coincide.

We take the self-referential approach to defining common knowledge by imposing the following three properties. First, the common knowledge of E implies the mutual knowledge of E: $C(E) \subseteq K_I(E)$. Second, if E is commonly known at ω, then there is a common basis F which is true at ω and which implies the very fact that E is common knowledge. Third, the event $C(E)$ is the largest event satisfying the above two properties.

Formally, we define the common knowledge operator $C : \mathcal{D} \to \mathcal{D}$ as follows.

$$C(E) := \max\{F \in \mathcal{J}(\mathcal{J}_I) \mid F \subseteq K_I(E)\} (\in \mathcal{J}(\mathcal{J}_I)) \text{ for each } E \in \mathcal{D}. \tag{5}$$

Note that it is $C(E) \in \mathcal{J}(\mathcal{J}_I)$ that ensures the second property (i.e., if $\omega \in C(E)$ then there is $F \in \mathcal{J}_I$ such that $\omega \in F \subseteq C(E)$). Note also that $C(E)$ is by definition a legitimate event.

In order to contrast common knowledge with mutual knowledge, we define the chain of mutual knowledge as follows. For a successor ordinal $\alpha = \beta + 1$, we define $K_I^\alpha := K_I \circ K_I^\beta$, starting with $K_I^1 := K_I$. For a limit ordinal $\alpha (> 0)$, we let $K_I^\alpha(\cdot) := \bigcap_{\beta : 1 \leq \beta < \alpha} K_I^\beta(\cdot)$. Generally, the mutual knowledge operator K_I^α is a well-defined operator for all $\alpha < \kappa$ in any κ knowledge space.

From now on, we ask three strands of questions. The first is the relation between individual, mutual, and common knowledge. The second is how our definition of the common knowledge operator relates to those posited in the previous literature. The third is how a knowledge morphism preserves common knowledge. We start with examining the sense in which common knowledge inherits logical and introspective properties of individual knowledge.

[50]First, as discussed in Footnote 4, we abuse the terminology, common knowledge (instead of common belief), even if individual players may violate Truth Axiom. Second, we can also introduce the common knowledge operator C_G among any subset of players $G \in \mathcal{P}(I)$, with the convention that $C_\emptyset := \mathrm{id}_{\mathcal{D}}$. Our analyses go through by replacing I with G. For example, recall the example in Footnote 18. There, player i's knowledge is represented by $K_{(0,i)}$ while her (non-probabilistic) belief is captured by $K_{(1,i)}$. In this case, we can consider the common knowledge among $\{0\} \times I$ and the common belief among $\{1\} \times I$.

Proposition 7 (Relation among Individual, Mutual, and Common Knowledge)**.** *Fix a κ-knowledge space $\overrightarrow{\Omega}$.*

1. *C always satisfies Positive Introspection.*

2. *If K_i satisfies No-Contradiction Axiom for some $i \in I$, then so does C. The same is true for Consistency and Truth Axiom.*

3. *If each K_i satisfies Monotonicity, then so does C.*

4. *Every K_i satisfies Necessitation iff C satisfies Necessitation.*

5. *If each K_i satisfies Monotonicity and Non-empty λ-Conjunction, then C satisfies Non-empty λ-Conjunction (as well as Monotonicity).*

6. *Negative Introspection of $(K_i)_{i \in I}$ does not necessarily imply that of C.*

7. *Let $\kappa = \infty$. Suppose that every K_i satisfies Truth Axiom, Positive Introspection, and Monotonicity. If each K_i also satisfies Negative Introspection, then C satisfies Negative Introspection.*

8. *Suppose that each K_i satisfies Positive Introspection and Monotonicity. Then, (i) C also satisfies Positive Introspection and Monotonicity; and (ii) for any knowledge operator $K : \mathcal{D} \to \mathcal{D}$ satisfying Positive Introspection and Monotonicity and satisfying $K(\cdot) \subseteq K_I(\cdot)$, we have $K(\cdot) \subseteq C(\cdot)$.*

9. *$C(\cdot) \subseteq K_I^{\alpha}(\cdot)$ for any ordinal $\alpha < \kappa$.*

The first eight statements of Proposition 7 demonstrate how the common knowledge operator does (and does not) inherit the properties of individual knowledge operators. The following are especially worth noting. First, Positive Introspection of C is endemic in its self-referential definition and the notion of common basis events. This holds regardless of assumptions on players' knowledge. In relation to this point, if an i' knowledge does not satisfy either Monotonicity or Positive Introspection, then i's knowledge is not necessarily the same as the common knowledge among $\{i\}$. See Remark A.5 in the Appendix for specific examples. They coincide with each other if both of Monotonicity and Positive Introspection are satisfied.

Second, while the common knowledge operator does not necessarily inherit Negative Introspection of individual knowledge operators, it does inherit Negative Introspection in an ∞-knowledge space satisfying Truth Axiom, Positive Introspection, and Monotonicity. This is because, in this case, the common knowledge is exactly the "individual" knowledge associated with the publicly evident events $\bigcap_{i \in I} \mathcal{J}_{K_i}$ and it inherits the closure under complementation, i.e., Negative Introspection.[51]

[51] Publicly evident events (Milgrom [60]) are also termed as self-evident events (Aumann [5] and Rubinstein and Wolinsky [72]), common truisms (Binmore and Brandenburger [11]), public events (Geanakoplos [31]), belief closed events (Lismont and Mongin [50]), evident knowledge events (Monderer and Samet [61]), common information (Nielsen [68]), and so forth.

Proposition 7 (8) reveals the relation between individual and common knowledge in terms of knowledgeability. Put differently, it clarifies the sense in which common knowledge has been understood as the "infimum" of players' knowledge.[52] For example, as is well known, common knowledge in a standard partitional model (Aumann [3]) is associated with the infimum partition (finest partition coarser than every player's partition). In this regard, Proposition 7 (7) also shows that the collection of publicly evident events $\bigcap_{i \in I} \mathcal{J}_{K_i}$ is exactly the infimum of players' knowledge in an ∞-knowledge space satisfying Truth Axiom, Positive Introspection, and Monotonicity. Indeed, for any κ-knowledge space satisfying Truth Axiom, Positive Introspection, and Monotonicity, the common knowledge is the infimum of players' knowledge because it is characterized by $\mathcal{J}(\bigcap_{i \in I} \mathcal{J}_{K_i})$. In other words, Proposition 7 (8) also holds for the combination of Truth Axiom, Positive Introspection, and Monotonicity.

Proposition 7 (9) means that the common knowledge of an event E implies the mutual knowledge of E up to any ordinal level $\alpha < \kappa$. Especially, if $\kappa \geq \aleph_1$ so that countable conjunction can legitimately be taken, the common knowledge of E implies the countable chain of mutual knowledge of E, which is often taken as an intuitive (or iterative) definition of common knowledge. That is, for a given κ-knowledge space with $\kappa \geq \aleph_1$, an event E is common knowledge, in an intuitive sense, within I at a state $\omega \in \Omega$ if, everybody (in I) knows E at ω, everybody knows that everybody knows E at ω, *ad infinitum*. We denote the intuitive notion of common knowledge of E among I by $K_I^\infty(E) := \bigcap_{n \in \mathbb{N}} K_I^n(E)$.[53] We will shortly ask when this intuitive notion coincides with the common knowledge operator defined by Equation (5).

The definition of common basis events plays an important role in establishing Proposition 7 (9) without imposing Monotonicity. McCarthy, Sato, Hayashi, and Igarashi [53] define the notion of what "any fool" knows in the sense that the knowledge of event by "any fool" implies the chain of mutual knowledge. In our definition, the notion of common knowledge can be interpreted as the individual knowledge associated with the common basis events.

Now, we go on to the second strand of questions. We start with showing that our

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[52] In this respect, we can formally introduce the notion of distributed knowledge as a dual of common knowledge, i.e., the "supremum" of players' knowledge, under Truth Axiom, Monotonicity, and Positive Introspection. Namely, we define the distributed knowledge operator as the knowledge operator associated with the collection $\sup_{i \in I} \mathcal{J}_i = \mathcal{J}(\bigcup_{i \in I} \mathcal{J}_{K_i})$: $D_I(E) := \max\{F \in \mathcal{J}(\bigcup_{i \in I} \mathcal{J}_{K_i}) \mid F \subseteq E\}$. By construction, D_I satisfies Truth Axiom, Positive Introspection, and Monotonicity. This definition captures the original idea by Halpern and Moses [34] that an event E is distributed knowledge at a state ω if someone who were informed of everything that each player $i \in I$ knows at ω would know E at ω.

[53] First, we denote by $\mathbb{N}$ the set of positive integers. That is, we do not require C to be correct. Second, we use the superscript ∞ instead of the least infinite ordinal (ω). The use of ∞ here is different from that in the context of ∞-complete algebras. Third, another intuitive (and weaker) notion of common knowledge would be that, an event E is common knowledge among I at ω, if, for any finite sequence of players $(i_1, i_2, \dots, i_n)$ in I with $n \in \mathbb{N}$, player i_1 knows that player i_2 knows that $\dots$ player i_n knows E at ω (i.e., $\omega \in (K_{i_n} \circ \cdots \circ K_{i_2} \circ K_{i_1})(E)$). It can be seen that, under Monotonicity and (Non-empty) λ-Conjunction with $\lambda > |I|$, these two intuitive notions coincide.

definition of common knowledge nests the previous characterizations when Monotonicity is assumed. Specifically, we consider the characterization by Monderer and Samet [61] in terms of publicly evident collections (Equation (6) below) and the "fixed-point" characterization by Friedell [29] and Halpern and Moses [34] (Equation (7) below).

Proposition 8 (Characterizations of Common Knowledge under Monotonicity)**.** *Let $\overrightarrow{\Omega}$ be a knowledge space such that every K_i satisfies Monotonicity. For each $E \in \mathcal{D}$,*

$$\begin{aligned} C(E) &= \{\omega \in \Omega \mid \textit{there is } F \in \bigcap_{i \in I} \mathcal{J}_{K_i} \textit{ with } \omega \in F \subseteq K_I(E)\} \\ &= \max\{F \in \mathcal{D} \mid F \in \bigcap_{i \in I} \mathcal{J}_{K_i} \textit{ and } F \subseteq K_I(E)\} \in \mathcal{D}; \textit{ and} \end{aligned} \tag{6}$$

$$C(E) = \max\{F \in \mathcal{D} \mid F = K_I(E) \cap K_I(F)\} (= \max\{F \in \mathcal{D} \mid F \subseteq K_I(E) \cap K_I(F)\}).$$

Suppose, in addition, that each K_i satisfies Finite Conjunction. For any $E \in \mathcal{D}$,

$$C(E) = \max\{F \in \mathcal{D} \mid F = K_I(E \cap F)\} (= \max\{F \in \mathcal{D} \mid F \subseteq K_I(E \cap F)\}). \tag{7}$$

Equation (6) follows because the common basis events coincide with the publicly evident events under Monotonicity. Thus, the last term in Equation (6) is well defined by the maximality property of $\mathcal{J}(\bigcap_{i \in I} \mathcal{J}_{K_i})$. In other words, C is the knowledge operator associated with $\mathcal{J}(\bigcap_{i \in I} \mathcal{J}_{K_i})$.

Monderer and Samet [61, Proposition] study the common knowledge of an event in a standard partitional knowledge space defined on the power set of underlying states of the world. On the other hand, the above characterization holds in any domain $\mathcal{D}$ and irrespective of assumptions on players' knowledge as long as Monotonicity is imposed. Likewise, the maximality property of $\mathcal{J}(\bigcap_{i \in I} \mathcal{J}_{K_i})$ ensures that Equation (7) be well defined.

Next, we ask when our notion of common knowledge coincides with the intuitive (iterative) notion. Imposing Monotonicity and Countable Conjunction on players' knowledge operators, the answer to the above question is to utilize a well-known (a variant of Tarski's) fixed point theorem stating that $K_I^\infty(E)$ is the greatest fixed point of the monotone and continuous set operator $f_E(F) = K_I(E \cap F)$ (see, for example, Halpern and Moses [34]). In other words, the intuitive definition ($C = K_I^\infty$), the fixed-point definition as well as the one by publicly evident events (Proposition 8), and our definition (Equation (5)) all agree as long as knowledge operators satisfy Monotonicity and Countable Conjunction.

Corollary 4 (Intuitive/Iterative Notion of Common Knowledge)**.** *Let $\overrightarrow{\Omega}$ be a κ-knowledge space where $\kappa \geq \aleph_1$ and knowledge operators satisfy Monotonicity and Countable Conjunction. Then, $C = K_I^\infty$.*

The reason that Corollary 4 may fail in the absence of Countable Conjunction is that K_I^∞ may fail to satisfy Positive Introspection $K_I^\infty(E) \subseteq K_I^\infty K_I^\infty(E)(= K_I^{\omega+\omega}(E))$ by the very fact that the chain of mutual knowledge stops at the least infinite ordinal level.[54]

We consider an implication of Corollary 4 when each player's knowledge satisfies the Kripke property and Positive Introspection. In this case, observe first that the "everyone-knows" operator K_I has the Kripke property as we have $b_{K_I}(\cdot) = \bigcup_{i \in I} b_{K_i}(\cdot)$. By induction, each mutual knowledge operator K_I^n has the Kripke property. Indeed, it satisfies that $b_{K_I^n}(\cdot) = b_{K_I}^n(\cdot)$, where $b_{K_I}^1 := b_{K_I}$ and, for $n \geq 2$, $b_{K_I}^n(\omega) := \bigcup_{\omega' \in b_{K_I}(\omega)} b_{K_I}^{n-1}(\omega')$ for each $\omega \in \Omega$. Then, it can be seen that the common knowledge operator $C = K_I^\infty$ inherits the Kripke property, where $b_C(\cdot) = \bigcup_{n \in \mathbb{N}} b_{K_I}^n(\cdot)$ (i.e., $b_C(\cdot)$ is equal to the transitive closure of $b_{K_I}(\cdot)$).

We can connect our argument to the notion of reachability (Aumann [3]).[55] We say that ω' is *reachable* from ω if there are sequences $(\omega^j)_{j=1}^m$ of Ω and $(i^j)_{j=1}^{m-1}$ of I with $m \in \mathbb{N} \setminus \{1\}$ such that $\omega = \omega^1$, $\omega' = \omega^m$, and $\omega^{j+1} \in b_{K_{i^j}}(\omega^j)$ for all $j \in \{1, \ldots, m-1\}$. Then, we have the following.

Proposition 9 (Common Knowledge and the Kripke Property/Reachability)**.** *Let $\overrightarrow{\Omega}$ be a κ-knowledge space with $\kappa \geq \aleph_1$ such that each player's knowledge satisfies the Kripke property and Positive Introspection. Then, each K_I^n (with $n \in \mathbb{N}$) and C inherit the Kripke property. Moreover, the following hold.*

1. *For each $n \in \mathbb{N}$, $b_{K_I^n} = b_{K_I}^n$.*
2. *$b_C(\omega) = \bigcup_{n \in \mathbb{N}} b_{K_I}^n(\omega) = \{\omega' \in \Omega \mid \omega' \text{ is reachable from } \omega\}$ for each $\omega \in \Omega$.*

As the third strand of questions, we ask when a knowledge morphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ preserves common knowledge in the following sense: for any event $E' \in \mathcal{D}'$, the event E' is common knowledge at $\varphi(\omega)$ iff $\varphi^{-1}(E')$ is common knowledge at ω'. Clearly, if the common knowledge operator can be written in terms of composites of individual knowledge operators, common knowledge is preserved under a knowledge morphism. For example, in the category of κ-knowledge spaces (with $\kappa \geq \aleph_1$) satisfying Monotonicity and Countable Conjunction, $C = \bigcap_{n \in \mathbb{N}} K_I^n$ (recall Corollary 4) commutes with the set-algebraic operations (i.e., $\bigcap_{n \in \mathbb{N}} K^n(\varphi^{-1}(E')) = \varphi^{-1}(\bigcap_{n \in \mathbb{N}} K'^n(E'))$).[56]

[54] While the first infinite ordinal number of chain of mutual knowledge is already hard to check in reality (Monderer and Samet [61]), the definition of common knowledge as a chain of mutual knowledge is in fact not strong enough to capture introspective properties of common knowledge (see, for example, Barwise [8] and Lismont and Mongin [50]).

[55] See also Halpern and Moses [34], Hérves-Beloso and Monteiro [42], and the references therein.

[56] On a related point, if $\kappa = \infty$ then Monotonicity guarantees that there is a limit ordinal α such that $K_I^\alpha = C$. Thus, common knowledge is preserved in such a case as well. See Remark A.6 in the Appendix.

Next, we examine how common knowledge is preserved between a given knowledge space and the universal knowledge space $\overrightarrow{\Omega^*}$ established in Section 3.1.[57]

Proposition 10 (Preservation of Common Knowledge in Universal Space)**.** *Suppose that assumptions on players' knowledge include Monotonicity and Positive Introspection and that they are homogeneously given across players in the given category of knowledge spaces. Then, for any knowledge space* $\overrightarrow{\Omega}$*, we have*

$$D_{\overrightarrow{\Omega}}^{-1}(C^*([e])) = C(D_{\overrightarrow{\Omega}}^{-1}([e])) \text{ for all } [e] \in \mathcal{D}^*,$$

where C *and* C^* *are the common knowledge operators in* $\overrightarrow{\Omega}$ *and* $\overrightarrow{\Omega^*}$*, respectively.*

We establish Proposition 10 in the following way. First, suppose that an event $[e]$ is commonly known at $D_{\overrightarrow{\Omega}}(\omega)$. Given that $\overrightarrow{\Omega}$ satisfies Monotonicity, it can be seen that $[\![e]\!]_{\overrightarrow{\Omega}} = D_{\overrightarrow{\Omega}}^{-1}([e])$ is commonly known at ω. For converse, it follows from the assumptions of the proposition that $C^* = K_i^*$ for all $i \in I$.

Generally, how does a knowledge morphism from one space to another preserve common knowledge? We provide a general characterization in the Appendix (Proposition A.4).

5 Knowledge and Common Knowledge of the Structure

Players' probabilistic beliefs are usually represented by the concept of types (Harsanyi [35]). Can we define a qualitative analogue of type mappings in order to capture players' knowledge?

In Section 5.1, we represent each player's knowledge at a state as a "knowledge-type" and show that this knowledge-type approach is equivalent to the knowledge operator approach for any given assumptions on players' knowledge. In Section 6, we establish a universal knowledge space in terms of hierarchies of knowledge-types (Theorem 3), and thus we formally connect Harsanyi's idea of representing players' probabilistic beliefs by belief-types to that of representing players' knowledge by knowledge-types.

The main goal of this section, however, is to formalize a notion of the structure of the model of knowledge by using players' knowledge-type mappings. We regard each player's knowledge-type mapping with a "signal" and ask the sense in which she knows hew own signal. Thus, in Section 5.2, we formally define what it means by a signal and examine the sense in which each player knows her knowledge-type mapping

[57] In Section 6, we also demonstrate the existence of a universal knowledge space in terms of hierarchies of knowledge. Proposition 10 also applies to the universal knowledge space established in Section 6.

in terms of her introspective properties. In Section 5.3, we apply this formalization to the universal knowledge space established in Section 3. By using this formalization, we (the outside analysts) can say that the players know (indeed, commonly know) the structure of the universal knowledge space.

5.1 Representing Knowledge by Knowledge-Types

Throughout the subsection, fix a κ-complete algebra $(\Omega, \mathcal{D})$, whee κ is an infinite (regular) cardinal or $\kappa = \infty$. Each knowledge-type is a mapping $\mu : \mathcal{D} \to \{0,1\}$ (i.e., $\mu \in \{0,1\}^{\mathcal{D}}$), where the knowledge of an event $E \in \mathcal{D}$ is captured by $\mu(E) = 1$. We represent each player's knowledge by a mapping, which we call a knowledge-type mapping, $t_i : \Omega \to \{0,1\}^{\mathcal{D}}$ with the following interpretation: player i knows an event E at state ω if $t_i(\omega)(E) = 1$ with $t_i(\omega)$ being her knowledge-type at ω.

Specifically, we define a knowledge-type mapping t_i in the following two steps. The first is to define the set of knowledge-types (a subset of $\{0,1\}^{\mathcal{D}}$) that reflects given assumptions on knowledge. Put differently, just as $\Delta(\cdot)$ stands for the set of σ-additive probability measures over a measurable space, we aim to formalize the set of legitimate binary "measures" $M(\cdot)$ which represents knowledge on a κ-complete algebra. The second is to define a knowledge-type mapping t_i as a mapping from $(\Omega, \mathcal{D})$ into $(M(\Omega, \mathcal{D}), \mathcal{M}(\Omega, \mathcal{D}))$ which satisfies the notion of knowledge in question, where $\mathcal{M}(\Omega, \mathcal{D})$ is a κ-complete algebra on $M(\Omega, \mathcal{D})$.

Our first task is to define $M(\Omega)(= M(\Omega, \mathcal{D}))$ as a given subset of $\{0,1\}^{\mathcal{D}}$ based on a concept of knowledge we adopt, as well as the κ-complete algebra $\mathcal{M}(\Omega, \mathcal{D})$. Recall that some properties of knowledge are referred to as logical properties (i.e., No-Contradiction Axiom, Consistency, Monotonicity, Non-empty λ-Conjunction with $\lambda \leq \kappa$, and Necessitation). The others are referred to as the introspective properties (Truth Axiom, Positive Introspection, and Negative Introspection) and the Kripke property.

We translate the logical properties of knowledge in terms of knowledge-types.

Definition 11 (Logical Properties of Knowledge-Types)**.** *Fix* $\mu \in \{0,1\}^{\mathcal{D}}$.

1. *No-Contradiction Axiom:* $\mu(\emptyset) = 0$.

2. *Consistency:* $\mu(E) \leq 1 - \mu(E^c)$ *for all* $E \in \mathcal{D}$.

3. *Monotonicity:* $\mu(E) \leq \mu(F)$ *for all* $E, F \in \mathcal{D}$ *with* $E \subseteq F$.

4. *Non-empty λ-Conjunction:* $\min_{E \in \mathcal{E}} \mu(E) \leq \mu(\bigcap \mathcal{E})$ *for all* $\mathcal{E} \subseteq \mathcal{D}$ *with* $0 < |\mathcal{E}| < \lambda \leq \kappa$.

5. *Necessitation:* $\mu(\Omega) = 1$.

The logical properties of μ are completely analogous to the corresponding logical properties of knowledge operators. Unlike (σ-)additive probabilities, however, it is possible that players do not know an event E nor its negation E^c at a particular state, and thus it is possible that $\mu(E) + \mu(E^c) = 0$.

Going back to the properties of knowledge operators, recall that properties of knowledge operators are related with each other. For example, Truth Axiom, which is referred to as an introspective property, implies the logical properties of No-Contradiction Axiom and Consistency. Thus, for a given concept of knowledge, fix a set of properties of knowledge to be assumed and consider all the implied logical properties. For example, if Truth Axiom, Monotonicity, and Negative Introspection are assumed, then we adopt all the logical properties as shown in Corollary 2 in Section 4.1, where λ-Conjunction becomes κ-Conjunction.

We define $M(\Omega, \mathcal{D})$ to be the set of mappings $\mu \in \{0,1\}^{\mathcal{D}}$ which satisfy all the implied logical properties in question. Thus, note that the set $M(\Omega, \mathcal{D})$ depends on properties of each player's knowledge we adopt. That is, while we suppress players' identities, $M(\Omega, \mathcal{D})$ depends on the identity of a player i if different assumptions on knowledge are assumed across players.

Next, we introduce a κ-complete algebra $\mathcal{M}(\mathcal{D})(= \mathcal{M}(\Omega, \mathcal{D}))$ on $M(\Omega)$. In an analogous way to the probabilistic type space approach (e.g., HS [39]), we define $\mathcal{M}(\mathcal{D})$ to be the κ-complete algebra generated by the sets of the form $\mathcal{M}_E := \{\mu \in M(\Omega) \mid \mu(E) = 1\}$ for all $E \in \mathcal{D}$. Abusing the notation, we sometimes write $\mathcal{M}(E) := \mathcal{M}_E$ when it is clear that E is not equal to the domain $\mathcal{D}$. Also, we denote by $\mathcal{M}^{(\Omega,\mathcal{D})}(E)$ if we stress the underlying κ-complete algebra $(\Omega, \mathcal{D})$. Thus, the κ-complete algebra that we introduce on $M(\Omega)$ is given by

$$\mathcal{M}(\mathcal{D}) := \mathcal{A}_\kappa(\{\mathcal{M}_E \in \mathcal{P}(M(\Omega)) \mid E \in \mathcal{D}\}).$$

The second task is to define a *knowledge-type mapping* as a κ-measurable mapping $t_i : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{M}(\mathcal{D}))$ which satisfies all the given properties of knowledge. This is done in the following three steps.

First, the above κ-measurablity condition of t_i means that the set $t_i^{-1}(\mathcal{M}_E)$ (i.e., the set of states at which player i knows an E) is an event. Formally, for all $E \in \mathcal{D}$,

$$t_i^{-1}(\mathcal{M}_E) = t_i^{-1}(\{\mu \in M(\Omega) \mid \mu(E) = 1\}) \in \mathcal{D}.$$

Second, we specify the logical properties of a knowledge-type mapping. A knowledge-type mapping t_i satisfies a given logical property iff $t_i(\omega)$ satisfies it for all $\omega \in \Omega$. Thus, for example, we say that t_i satisfies *No-Contradiction Axiom* if $t_i(\omega)$ satisfies it (i.e., $t_i(\omega)(\emptyset) = 0$) for all $\omega \in \Omega$. The other logical properties (Consistency, Monotonicity, Non-empty λ-Conjunction, and Necessitation) are defined in the same way.

Third, we define below the introspective and Kripke properties of a knowledge-type mapping. Then, a knowledge-type mapping is formally defined as a κ-measurable mapping $t_i : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{M}(\mathcal{D}))$ which satisfies the required properties of knowledge.

Definition 12 (Introspective Properties of Knowledge-Types)**.** *Fix a κ-measurable mapping $t_i : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{M}(\mathcal{D}))$. We define the introspective and Kripke properties of t_i as follows.*

1. *Truth Axiom: for any $\omega \in \Omega$ and $E \in \mathcal{D}$, $t_i(\omega)(E) = 1$ implies $\omega \in E$.*
2. *Positive Introspection: for any $\omega \in \Omega$ and $E \in \mathcal{D}$, $t_i(\omega)(E) = 1$ implies $t_i(\omega)(\{\omega' \in \Omega \mid t_i(\omega')(E) = 1\}) = 1$ (i.e., $t_i(\omega)(t_i^{-1}(\mathcal{M}_E)) = 1$).*
3. *Negative Introspection: for any $\omega \in \Omega$ and $E \in \mathcal{D}$, $t_i(\omega)(E) = 0$ implies $t_i(\omega)(\{\omega' \in \Omega \mid t_i(\omega')(E) = 0\}) = 1$ (i.e., $t_i(\omega)(\neg t_i^{-1}(\mathcal{M}_E)) = 1$).*
4. *The Kripke property: for any $\omega \in \Omega$ and $E \in \mathcal{D}$, $b_{t_i}(\omega)(:= \bigcap\{F \in \mathcal{D} \mid t_i(\omega)(F) = 1\}) \subseteq E$ implies $t_i(\omega)(E) = 1$.*

We call a κ-measurable mapping $t_i : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{M}(\mathcal{D}))$ to be a knowledge-type mapping if it satisfies all the required properties of knowledge in question.

In order to conclude this subsection, we establish that this knowledge-type approach is equivalent to the knowledge operator approach. Fix a notion of knowledge (i.e., a category of knowledge spaces). Suppose that each player's knowledge is assigned by her knowledge operator $K_i : \mathcal{D} \to \mathcal{D}$. Let $M(\Omega, \mathcal{D})$ be the set of i's knowledge-types which satisfy the logical properties of knowledge derived from i's knowledge operators of the given category (note again that we suppress players' identities on $M(\Omega)$).

Now, let $t_{K_i} : \Omega \to M(\Omega)$ be such that for each $\omega \in \Omega$ and $E \in \mathcal{D}$,

$$t_{K_i}(\omega)(E) = \begin{cases} 1 & \text{if } \omega \in K_i(E) \\ 0 & \text{otherwise} \end{cases}.$$

The mapping t_{K_i} clearly inherits all the properties of K_i, and hence it is well defined as a map $t_{K_i} : \Omega \to M(\Omega)$.[58] We also have $t_{K_i}^{-1}(\mathcal{M}_E) = K_i(E) \in \mathcal{D}$, so that $t_{K_i} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{M}(\mathcal{D}))$ is κ-measurable. Hence, t_{K_i} is a knowledge-type mapping.

Conversely, suppose that a knowledge-type mapping $t_i : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{M}(\mathcal{D}))$ is given. Then, we define the knowledge operator $K_{t_i} : \mathcal{D} \to \mathcal{D}$ as follows:

$$K_{t_i}(E) := \{\omega \in \Omega \mid t_i(\omega)(E) = 1\} (= t_i^{-1}(\mathcal{M}_E)) \in \mathcal{D} \text{ for each } E \in \mathcal{D}.$$

It is the κ-measurability of t_i that ensures that K_{t_i} send events to events. It can be easily seen that K_{t_i} inherits all the properties imposed on t_i. Again, see Remark A.7.

[58]For completeness, we provide a proof in Remark A.7 in the Appendix.

Finally, it is clear that $K_{t_{K_i}} = K_i$ and $t_{K_{t_i}} = t_i$. In fact, given K_i, we have $\omega \in K_{t_{K_i}}(E)$ iff $t_{K_i}(\omega)(E) = 1$ iff $\omega \in K_i(E)$. Given t_i, we have $t_{K_{t_i}}(\omega)(E) = 1$ iff $\omega \in K_{t_i}(E)$ iff $t_i(\omega)(E) = 1$. The above equalities suggest that a knowledge space is equivalently defined by $\langle(\Omega, \mathcal{D}), (t_i)_{i\in I}, \Theta\rangle$, where each $t_i : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{M}(\mathcal{D}))$ is i's knowledge-type mapping.[59]

To conclude this subsection, we have two remarks. First, we can define the possibility-type mapping $\overline{t}_i$ by $\overline{t}_i(\omega)(E) := 1 - t_i(\omega)(E^c)$ in light of the possibility operator induced from a knowledge operator. It is clear that specifying possibility-types is equivalent to specifying knowledge-types. Using the notion of possibility-types, for example, we can re-write Consistency as $t_i(\omega)(\cdot) \leq \overline{t}_i(\omega)(\cdot)$ for all $\omega \in \Omega$.

Second, specifying an i's knowledge-type at each state is equivalent to specifying the collection of events that i knows at that state.[60] For example, for a given knowledge-type mapping t_i, the collection of events that i knows at ω is given by $(t_i(\omega))^{-1}(\{1\}) = \{E \in \mathcal{D} \mid t_i(\omega)(E) = 1\}$.

5.2 Meta-Knowledge of Knowledge-Type Mappings

In a given knowledge space, the objects of knowledge are events, i.e., a subset of states of the world. How can we extend the notion of the knowledge of an event to that of a mapping such as players' strategies defined on the states of the world?

In this subsection, we first define a notion of a signal mapping. It is simply a mapping defined on underlying states of the world. We formalize the idea that a player knows a signal mapping. Examples of signal mappings include action/decision, strategies, random variables, and (knowledge-)type mappings. Second, we define a notion of informativeness derived from a signal mapping. Third, we apply these notions to players' type mappings in order to study the meta-knowledge of the structure of a model.[61] Especially, we examine what it means by the fact that a player knows her knowledge-type mapping.

Throughout the subsection, $(\Omega, \mathcal{D})$ refers to a κ-complete algebra, where κ is an infinite (regular) cardinal or $\kappa = \infty$. We start with defining a notion of a signal mapping. Let X be a set and let $\mathcal{D}_X$ be a subset of $\mathcal{P}(X)$. A *signal* mapping is a mapping $x : (\Omega, \mathcal{D}) \to (X, \mathcal{D}_X)$ satisfying $x^{-1}(\mathcal{D}_X) \subseteq \mathcal{D}$. Examples include strategies, action/decision functions, random variables, and so on. Now, we define a notion that an agent knows a signal mapping.

[59] Formally, the category of knowledge spaces in terms of knowledge operators is equivalent (in the category theoretical sense) to that of knowledge spaces in terms of knowledge-type mappings. See Remark A.8 in the Appendix for the proof.

[60] This latter specification corresponds to a neighborhood (Montague-Scott) system in the context of modal logic.

[61] See Footnote 35 for the literature posing the meta-knowledge of the structure itself. Bacharach [6, 7] formalizes the event that an agent has an information partition by regarding it as a signal mapping in a related manner.

Definition 13 (Knowledge of Signal)**.** *Let $\overrightarrow{\Omega}$ be a knowledge space, and let $x : (\Omega, \mathcal{D}) \to (X, \mathcal{D}_X)$ be a signal mapping. An agent i is said to know the signal mapping $x : (\Omega, \mathcal{D}) \to (X, \mathcal{D}_X)$ at ω (or, she knows the mapping x at ω with respect to $\mathcal{D}_X$) if $x^{-1}(E_X) \subseteq K_i(x^{-1}(E_X))$ for any $E_X \in \mathcal{D}_X$ with $x(\omega) \in E_X$. If i knows the signal mapping x at any state (equivalently, if $x^{-1}(\mathcal{D}_X) \subseteq \mathcal{J}_{K_i}$), then we say that i knows x.*

We have the following four remarks. First, at an interpretational level, when we examine whether an agent i knows a signal mapping $x : (\Omega, \mathcal{D}) \to (X, \mathcal{D}_X)$, we pre-suppose that the mapping x is observable to i with respect to $\mathcal{D}_X$.

Second, in words, an agent i knows a signal mapping $x : (\Omega, \mathcal{D}) \to (X, \mathcal{D}_X)$ at ω when any event $x^{-1}(E_X)$ containing ω is self-evident to i. If $\mathcal{D}_X$ contains the singleton $\{x(\omega)\}$ and if the agent i knows the signal mapping x at ω, then $[x(\omega)] := x^{-1}(\{x(\omega)\}) = \{\omega' \in \Omega \mid x(\omega') = x(\omega)\}$ is self-evident to her.[62]

Third, the knowledge of the signal mapping x is considered to be the "measurability" of $x : (\Omega, \mathcal{J}_{K_i}) \to (X, \mathcal{D}_X)$.[63] Indeed, recall that $\mathcal{J}_{K_i}$ forms a (κ-complete) sub-algebra of $\mathcal{D}$ if i's knowledge satisfies Truth Axiom, Monotonicity, and Negative Introspection.

Fourth, we can extend the notion of a signal mapping to common knowledge. We say that $x : (\Omega, \mathcal{D}) \to (X, \mathcal{D}_X)$ is *common knowledge* at ω (or, x is *common knowledge* at ω with respect to $\mathcal{D}_X$) if, for any $E_X \in \mathcal{D}_X$ with $x(\omega) \in E_X$, we have $x^{-1}(E_X) \subseteq C(x^{-1}(E_X))$. Next, we say that x is *common knowledge* if it is common knowledge at any state (i.e., $x^{-1}(E_X) \subseteq C(x^{-1}(E_X))$ for all $E_X \in \mathcal{D}_X$).

We proceed with defining a concept of informativeness derived from a signal mapping.

Definition 14 (Informativeness according to Signal)**.** *Fix states ω and ω' in Ω. We say that ω is at least as informative as ω' according to a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{D}_X)$ if*

$$\{E_X \in \mathcal{D}_X \mid \omega' \in x^{-1}(E_X)\} \subseteq \{E_X \in \mathcal{D}_X \mid \omega \in x^{-1}(E_X)\}.$$

Likewise, we say that states ω and ω' are equally informative according to $x : (\Omega, \mathcal{D}) \to (X, \mathcal{D}_X)$ if

$$\{E_X \in \mathcal{D}_X \mid \omega' \in x^{-1}(E_X)\} = \{E_X \in \mathcal{D}_X \mid \omega \in x^{-1}(E_X)\}.$$

The ideas behind Definition 14 are (i) that the informational content of a signal mapping $x : (\Omega, \mathcal{D}) \to (X, \mathcal{D}_X)$ at ω is expressed as the collection of sets $\{E_X \in$

[62] In the literature (e.g., the one on a characterization of a solution concept of a game in state space models of knowledge), the knowledge of a mapping $x : (\Omega, \mathcal{D}) \to X$ at ω is often defined by the requirement that $[x(\omega)]$ is self-evident (e.g., BDG [17] and Geanakoplos [31]). This corresponds to the case where $\mathcal{D}_X = \{[x(\omega)] \in \mathcal{P}(X) \mid \omega \in \Omega\}$ (provided that $x^{-1}(\mathcal{D}_X) \subseteq \mathcal{D}$). If, for example, x is an action function, then $\mathcal{D}_X = \{[x(\omega)] \mid \omega \in \Omega\}$ is associated with actions that could have been taken at each state.

[63] For example, Aumann [4] defines the knowledge of a strategy by its measurability with respect to a partition.

$\mathcal{D}_X \mid x(\omega) \in E_X\}$ and (ii) that informational contents are ranked by the implication in the form of set inclusion.[64] The notion of informativeness is clearly reflexive and transitive.

For the rest of this subsection, we apply these notions to knowledge-type mappings. Thus, we firstly examine the fact that a player knows her knowledge-type mapping in terms of introspection. Second, we study the relation between informativeness and possibility in order to characterize the sense in which a player knows her knowledge-type mapping in terms of informativeness.

The first aim is thus to characterize what it means by the fact that player i knows her knowledge-type mapping, in terms of introspection.

Proposition 11 (Knowledge of Type and Introspection)**.** *Let* $\overrightarrow{\Omega} := \langle (\Omega, \mathcal{D}), (t_i)_{i \in I}, \Theta \rangle$ *be a knowledge space. Fix* $i \in I$*.*

1. *Player* i *knows her knowledge-type mapping* t_i *with respect to* $\{\mathcal{M}(E) \mid E \in \mathcal{D}\}$ *iff* $K_{t_i}(\cdot) \subseteq K_{t_i} K_{t_i}(\cdot)$.
2. *Player* i *knows* t_i *with respect to* $\{\neg\mathcal{M}(E) \mid E \in \mathcal{D}\}$ *iff* $(\neg K_{t_i})(\cdot) \subseteq K_{t_i}(\neg K_{t_i})(\cdot)$.

We make the following three remarks. First, in the literature of (standard) non-partitional models of knowledge, the lack of Negative Introspection is interpreted as the fact that an agent does not know her own possibility correspondence.[65] Without imposing Negative Introspection, an agent does not know her own knowledge-type mapping with respect to $\{\mathcal{M}(E) \mid E \in \mathcal{D}\} \cup \{\neg\mathcal{M}(E) \mid E \in \mathcal{D}\}$. Rather, she takes her own information at face value in the sense that she only knows her knowledge-type mapping with respect to her own knowledge (i.e., $\{\mathcal{M}(E) \mid E \in \mathcal{D}\}$).

Second, it is worth pointing out that Positive Introspection and Negative Introspection in Proposition 11 pertain to every event including $E = K_j(F)$ for some $F \in \mathcal{D}$. This means that if player i knows her knowledge-type mapping t_i with respect to $\{\mathcal{M}(E) \mid E \in \mathcal{D}\}$ then she also knows such knowledge-type mapping as $t_{K_{t_i} K_{t_j}}$ with respect to $\{\mathcal{M}(E) \mid E \in \mathcal{D}\}$, where $t_{K_{t_i} K_{t_j}}$ is the knowledge-type mapping associated with the operator $K_{t_i} K_{t_j}$ (i.e., $t_{K_{t_i} K_{t_j}}(\omega)(E) = 1$ iff $\omega \in K_{t_i} K_{t_j}(E)$). On the one hand, the fact that each i knows $t_{K_{t_i} K_{t_j}}$ could possibly be a justification for why we (the outside analysts) can assume that "i knows j's knowledge operator in i's mind." On the other hand, it is implicitly assumed that player i figures out what K_{t_j} is. We'll revisit (in Proposition 13 in Section 5.3) the question whether we (the outside analysts) can assume that each player knows each other's knowledge-type mapping in relation to a universal knowledge space.

Third, we can also characterize Positive Introspection and Negative Introspection in terms of player i's possibility correspondence $b_i : (\Omega, \mathcal{D}) \to (\mathcal{P}(\Omega), \{\{F \in \mathcal{D} \mid F \subseteq$

[64] The notion of informativeness is closely related to that of information studied by Bonanno [13]. Lipman [49] and Mukerji [67] also study the idea that informational contents are ranked by the implication in the form of set inclusion.

[65] See, for instance, BDG [17], Dekel and Gul [20], and Geanakoplos [31].

$E\} \mid E \in \mathcal{D}\})$ (where b_i is induced from either t_i or K_i) if i's knowledge satisfies the Kripke property (thus, b_i satisfies the regularity condition that $b_i^{-1}(\{F \in \mathcal{D} \mid F \subseteq E\}) \in \mathcal{D}$ for all $E \in \mathcal{D}$). Then, i's knowledge satisfies Positive Introspection iff i knows her possibility correspondence b_i with respect to $\{\{F \in \mathcal{D} \mid F \subseteq E\} \mid E \in \mathcal{D}\}$. Likewise, i's knowledge satisfies Negative Introspection iff i knows b_i with respect to $\{\{F \in \mathcal{D} \mid E^c \cap F \neq \emptyset\} \mid E \in \mathcal{D}\}$.

We go on to the second aim. Namely, we apply the notion of informativeness to i's knowledge-type mapping $t_i : (\Omega, \mathcal{D}) \to (M(\Omega), \{\mathcal{M}(E) \mid E \in \mathcal{D}\})$. Observe the following. For states ω and ω' in Ω, ω is at least as informative as ω' to i (precisely, according to $t_i : (\Omega, \mathcal{D}) \to (M(\Omega), \{\mathcal{M}(E) \mid E \in \mathcal{D}\})$) iff $t_i(\omega')(E) \leq t_i(\omega)(E)$ for all $E \in \mathcal{D}$. Likewise, states ω and ω' are equally informative according to i iff $t_i(\omega) = t_i(\omega')$.

We define the set of states that are at least as informative to i as ω:

$$(\uparrow t_i(\omega)) = \{\omega' \in \Omega \mid t_i(\omega)(E) \leq t_i(\omega')(E) \text{ for all } E \in \mathcal{D}\}.$$

We also define, for each $\omega \in \Omega$,

$$\begin{aligned}(\downarrow t_i(\omega)) &= \{\omega' \in \Omega \mid t_i(\omega')(E) \leq t_i(\omega)(E) \text{ for all } E \in \mathcal{D}\} \text{ and} \\ [t_i(\omega)] &= \{\omega' \in \Omega \mid t_i(\omega) = t_i(\omega')\}(= (\uparrow t_i(\omega)) \cap (\downarrow t_i(\omega))).\end{aligned}$$

If $\omega' \in [t_i(\omega)]$ then ω and ω' are indistinguishable to player i in the sense that her knowledge-types (and thus the collections of events that she knows) are exactly the same at these states. Put differently, the equal informativeness is translated into the indistinguishability. Thus, the collection $\{[t_i(\omega)] \mid \omega \in \Omega\}$ forms a partition of Ω generated by the knowledge-type mapping t_i. Note that $(\uparrow t_i(\omega))$, $(\downarrow t_i(\omega))$, and $[t_i(\omega)]$ may not necessarily be events.[66]

As a remark, we mention that if ω' is at least as informative to i as ω (i.e., $\omega' \in (\uparrow t_i(\omega))$), then we have

$$b_{t_i}(\omega') = \bigcap\{E \in \mathcal{D} \mid t_i(\omega')(E) = 1\} \subseteq \bigcap\{E \in \mathcal{D} \mid t_i(\omega)(E) = 1\} = b_{t_i}(\omega).$$

That is, the informativeness relation implies the set-inclusion between possibility sets. Note that if t_i satisfies the Kripke property, then the converse also holds: $b_{t_i}(\omega') \subseteq b_{t_i}(\omega)$ implies $\omega' \in (\uparrow t_i(\omega))$. This is simply because, if $t_i(\omega)(E) = 1$ then $b_{t_i}(\omega') \subseteq b_{t_i}(\omega) \subseteq E$ and thus $t_i(\omega')(E) = 1$. In other words, if the Kripke property is met, then the possibility set can be alternatively used to define the notion of informativeness.

Now, we examine the sense in which a player knows her knowledge-type mapping by studying how introspective properties imply the relations between informativeness and possibility.

[66]Suppose that $\mathcal{D}$ is a complete algebra. Since $\{\mu\} = \bigcap\{\mathcal{M}_E \in \mathcal{M}(\mathcal{D}) \mid \mu \in \mathcal{M}_E \text{ and } E \in \mathcal{D}\} \in \mathcal{M}(\mathcal{D})$, it follows that $\mathcal{M}(\mathcal{D}) = \mathcal{P}(M(\Omega, \mathcal{D}))$. Now, it is always the case that $(\uparrow t_i(\omega)) = t_i^{-1}(\{\mu \in M(\Omega) \mid t_i(\omega)(E) \leq \mu(E) \text{ for all } E \in \mathcal{D}\}) \in \mathcal{D}$, $(\downarrow t_i(\omega)) = t_i^{-1}(\{\mu \in M(\Omega) \mid \mu(E) \leq t_i(\omega)(E) \text{ for all } E \in \mathcal{D}\}) \in \mathcal{D}$, and $[t_i(\omega)] = t_i^{-1}(\{t_i(\omega)\}) \in \mathcal{D}$ for all $\omega \in \Omega$.

Proposition 12 (Relation between Possibility and Informativeness)**.** *Let $\overrightarrow{\Omega}$ be a knowledge space.*

1. *t_i satisfies Truth Axiom iff $(\omega \in [t_i(\omega)] \subseteq)(\uparrow t_i(\omega)) \subseteq b_{t_i}(\omega)$ (for all $\omega \in \Omega$).*

2. *If t_i satisfies Positive Introspection, then $b_{t_i}(\omega) \subseteq (\uparrow t_i(\omega))$. If t_i satisfies the Kripke property, the converse is also true.*

3. *If t_i satisfies Negative Introspection, then $b_{t_i}(\omega) \subseteq (\downarrow t_i(\omega))$. If t_i satisfies the Kripke property, the converse is also true.*

4. *If t_i satisfies Truth Axiom, (Positive Introspection), and Negative Introspection, then $(\uparrow t_i(\omega)) = (\downarrow t_i(\omega)) = [t_i(\omega)] = b_{t_i}(\omega)$. If t_i satisfies the Kripke property, the converse is also true.*

First, when player i's knowledge-type mapping satisfies Truth Axiom, informativeness implies possibility. Conversely, when t_i satisfies Positive Introspection, possibility implies informativeness. Hence, when player i's knowledge-type mapping satisfies Truth Axiom and Positive Introspection, the notions of informativeness and possibility coincide. A simple corollary of this argument is that, as with standard possibility correspondence models, if t_i satisfies Truth Axiom and Positive Introspection as well as the Kripke property, then b_{t_i} is reflexive and transitive.[67]

Second, when player i's knowledge-type mapping satisfies Truth Axiom, Positive Introspection, and Negative Introspection, then either notion of informativeness or possibility induces the same informational partition $\{b_{t_i}(\omega) \mid \omega \in \Omega\} = \{[t_i(\omega)] \mid \omega \in \Omega\}$ of Ω with the following property: what she knows at ω coincides with that at ω' for any $\omega' \in [t_i(\omega)] = b_{t_i}(\omega)$.

Finally, we mention that, under Truth Axiom and Positive Introspection, we can define the notion of possibility between states in terms of a self-evident collection as $b_{t_i}(\omega) = \bigcap\{E \in \mathcal{J}_{t_i} \mid \omega \in E\}$, where $\mathcal{J}_{t_i} := \{E \in \mathcal{D} \mid t_i(\omega)(E) = 1 \text{ for any } \omega \in E\}$. This follows because $\mathcal{J}_{t_i} = \{K_{t_i}(E) \mid E \in \mathcal{D}\}$ under Truth Axiom and Positive Introspection. If $\kappa = \infty$ and if player i's knowledge satisfies Arbitrary Conjunction and Monotonicity, then $b_{t_i}(\omega)$ is indeed the minimal self-evident event containing ω. Furthermore, the notion of common knowledge is characterized by $\bigcap_{i \in I} \mathcal{J}_{t_i}$ in this special case. Thus the possibility correspondence associated with common knowledge is simply given by $\bigcap\{E \in \bigcap_{i \in I} \mathcal{J}_{t_i} \mid \omega \in E\}$ in this special case.

5.3 How Do the Players Know the Structure of a Universal Knowledge Space?

As we have discussed at the end of Section 3.1, we ask the sense in which the players know (or commonly know) the structure of the model itself.[68] Consider the universal

[67] See the references cited in Footnote 3 for such reflexive and transitive (non-partitional) possibility correspondence models.

[68] For the literature, see Footnote 35.

knowledge space $\overrightarrow{\Omega}^*$ established in Section 3.1. We apply the notion of the knowledge of signal mappings to the knowledge-type mappings associated with each player's knowledge operator K_i^* and the common knowledge operator C^*.[69]

Proposition 13 (Do Players Know the Model of Knowledge?)**.** *Suppose that, in the given category of knowledge spaces, assumptions on players' knowledge include Monotonicity and Positive Introspection and that they are homogeneously given across players. Consider the knowledge-type mappings* $t_{K_i^*} : (\Omega, \mathcal{D}) \to (M(\Omega^*, \mathcal{D}^*), \{\mathcal{M}([e]) \mid [e] \in \mathcal{D}^*\})$ *and* $t_{C^*} : (\Omega, \mathcal{D}) \to (M(\Omega^*, \mathcal{D}^*), \{\mathcal{M}([e]) \mid [e] \in \mathcal{D}^*\})$*. Each mapping is commonly known among* I*.*

We establish Proposition 13 in the following way. First, if assumptions on players' knowledge are homogeneous, then every player's knowledge-type mapping is identical in the universal knowledge space $\overrightarrow{\Omega^*}$. Under Positive Introspection and Monotonicity, every player's knowledge-type mapping is also identical with the type mapping associated with the common knowledge operator. Second, Positive Introspection enables each player to know their own type mappings (see Proposition 11). But since their type mappings are homogeneous, it follows that every player know every player's type mapping. Indeed, players' knowledge-type mappings (and the knowledge-type mapping associated with their common knowledge) are commonly known.

The last step in the above reasoning, however, is worth scrutinizing. First, in concluding that the knowledge-type mappings are commonly known, we (i.e., the outside analysts) use the exogenous assumption that each player's knowledge is homogeneously given. Thus, the above proposition renders the sense in which we (i.e., the analysts) can conclude that the type-mappings are commonly known among the players within the model. Second, in order for us (i.e., the outside analysts) to ask i's knowledge about j's knowledge-type mapping, it is an important underlying assumption that assumptions on players' knowledge are homogeneously given. This is because j's knowledge-type mapping $t_{K_j^*} : (\Omega, \mathcal{D}) \to (M(\Omega^*, \mathcal{D}^*), \{\mathcal{M}([e]) \mid [e] \in \mathcal{D}^*\})$ would be unobservable to player i without such an exogenous assumption.

Also, note that Proposition 13 does not necessarily claim that players' knowledge-type mappings are commonly known with respect to $\{\mathcal{M}([e]) \mid [e] \in \mathcal{D}^*\} \cup \{\neg\mathcal{M}([e]) \mid [e] \in \mathcal{D}^*\}$ unless otherwise each player's knowledge satisfies Negative Introspection. Indeed, if some player fails to satisfy Negative Introspection, she does not know her knowledge-type mapping with respect to $\{\mathcal{M}([e]) \mid [e] \in \mathcal{D}^*\} \cup \{\neg\mathcal{M}([e]) \mid [e] \in \mathcal{D}^*\}$.

Finally, we make the following two remarks. First, it is clear (from Lemma 3 and Proposition 7) that if every player's knowledge is assumed to satisfy Necessitation then the common knowledge operator satisfies Necessitation (i.e., $\Omega^* = C(\Omega^*)$). Thus, any valid expression is commonly known among the players.

[69] In Section 6, we establish a universal knowledge space in terms of hierarchies of knowledge-types. There, the knowledge-type mappings are a primitive of the knowledge space, and thus we can directly apply our discussion (instead of defining the knowledge-type mappings $t_{K_i^*}$ from K_i^*).

Second, consider a category of knowledge spaces which satisfies the assumptions imposed in Proposition 13. Take any knowledge space $\overrightarrow{\Omega}$ in the given category. Then, as we have seen in Remark 2, $\overrightarrow{D(\Omega)}$ is a knowledge subspace of $\overrightarrow{\Omega^*}$. The statements of Proposition 13 hold with respect to the knowledge subspace $\overrightarrow{D(\Omega)}$. That is, for any knowledge space $\overrightarrow{\Omega}$, players' type mappings are commonly known in $\overrightarrow{D(\Omega)}$. In contrast, players' type mappings may not be necessarily commonly known in $\overrightarrow{\Omega}$. This is because players' knowledge may be different across players in the given space $\overrightarrow{\Omega}$. See Proposition A.5 of the Appendix.

6 A Hierarchical Approach to a Universal Knowledge Space

Here, we provide an alternative construction of a universal knowledge space by representing players' interactive knowledge as hierarchies of knowledge-types instead of infinitary languages. In this section, each player's knowledge in a given knowledge space is given by her knowledge-type mapping.

Specifically, we follow HS's [39] hierarchical approach to establishing a universal type space.[70] Before we go on to the construction, we make the following two preliminary remarks. First, we define a product κ-complete algebra as follows.

Remark 3 (Product κ-Complete Algebra)**.** Let $(\Omega_j, \mathcal{D}_j)$ be a κ-complete algebra for each $j \in J$, where J is a non-empty index set. We define the product κ-complete algebra $\prod_{j \in J} \mathcal{D}_j$ on the product space $\prod_{j \in J} \Omega_j$ as

$$\prod_{j \in J} \mathcal{D}_j := \mathcal{A}_\kappa \left(\bigcup_{j \in J} \{ \pi_j^{-1}(E) \mid E \in \mathcal{D}_j \} \right),$$

where $\pi_j : \prod_{j \in J} \Omega_j \to \Omega_j$ is the projection for each $j \in J$. As usual, if $\mathcal{D} = \mathcal{D}_j$ for all $j \in J$, then we denote $\prod_{j \in J} \mathcal{D}_j$ by $\mathcal{D}^J$. We also define finite products such as $\mathcal{D}_1 \times \mathcal{D}_2 = \prod_{j \in \{1,2\}} \mathcal{D}_j$ in the usual manner.

Second, we express knowledge morphisms in terms of knowledge-type mappings, which is analogous to the one in the literature on the (belief-)type spaces.

Remark 4 (Knowledge Morphism in terms of Knowledge-Tye Mapping)**.** Let $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$ be knowledge spaces. Let $\varphi : \Omega \to \Omega'$ be a mapping. Condition (3) in Definition 3 can be written as follows.

$$t'_i(\varphi(\omega))(E') = t_i(\omega)(\varphi^{-1}(E')) \text{ for all } \omega \in \Omega \text{ and } E' \in \mathcal{D}'.$$

[70] First, other hierarchical representations of knowledge include Fagin [25], FGHV [26], FHV [28], and HS [40, 41] in related contexts. Second, Viglizzo [83] studies HS's [39] hierarchical approach as an alternative construction of Moss and Viglizzo's [65, 66] terminal (final) coalgebra for a measure polynomial functor.

Throughout the section, fix an infinite regular cardinal κ, a κ-complete algebra of states of nature $(S, \mathcal{A}_S)$, and a non-empty set of players I. We also fix assumptions on players' knowledge.

6.1 A Hierarchical Construction of a Universal Knowledge Space

We proceed with constructing a universal knowledge space in terms of hierarchies in four steps. The first step is to define the hierarchies space, which encompasses all the hierarchies of players' interactive knowledge regarding $(S, \mathcal{A}_S)$ up the ordinality of κ.

Definition 15 (Hierarchies Space)**.** *We define the sequence of κ-complete algebras $(H^\alpha, \mathcal{H}^\alpha)$ for ordinals $\alpha \leq \kappa$ as follows.*

1. *For $\alpha = 0$, we let $(H^0, \mathcal{H}^0) := (S, \mathcal{A}_S)$.*
2. *For any ordinal α with $0 < \alpha < \kappa$, we let*
$$(H^\alpha, \mathcal{H}^\alpha) := \left(S \times \prod_{\beta<\alpha} M(H^\beta, \mathcal{H}^\beta)^I, \mathcal{A}_S \times \prod_{\beta<\alpha} \mathcal{M}(H^\beta, \mathcal{H}^\beta)^I \right).$$
3. *For $\alpha = \kappa$, we define $(H^\kappa, \mathcal{H}^\kappa)$ by $H^\kappa := S \times \prod_{\alpha<\kappa} M(H^\alpha, \mathcal{H}^\alpha)^I$ and*
$$\mathcal{H}^\kappa := \{(\pi^{\kappa,\alpha})^{-1}(E^\alpha) \mid E^\alpha \in \mathcal{H}^\alpha \text{ for some } \alpha < \kappa\},$$
where $\pi^{\alpha,\beta} : H^\alpha \to H^\beta$ is the projection for all ordinals (α, β) with $0 \leq \beta \leq \alpha \leq \kappa$.

When $\alpha = \gamma$, we omit the superscript κ. Thus, we denote $(H, \mathcal{H}) := (H^\kappa, \mathcal{H}^\kappa)$. We remark that $(H, \mathcal{H})$ is indeed a κ-complete algebra (see Remark A.9 in the Appendix).

We often identify the hierarchies space H with the product of the set of states of nature S and each player's hierarchies space. To that end, for each $i \in I$ and α with $1 \leq \alpha \leq \kappa$, we let $H_i^\alpha := \prod_{\beta<\alpha} M(H^\beta, \mathcal{H}^\beta)$. Then, for any ordinal $\alpha \geq 1$, we can identify
$$H^\alpha = S \times \prod_{i \in I} H_i^\alpha. \tag{8}$$
Again, we omit the superscript κ when $\alpha = \kappa$. For all ordinals (α, β) with $1 \leq \beta \leq \alpha \leq \kappa$, we denote by $\pi_i^{\alpha,\beta} : H_i^\alpha \to H_i^\beta$ the projection.

In the second step, we define descriptions in terms of hierarchies. For any given knowledge space $\overrightarrow{\Omega}$, we define the (hierarchical) description map $h : \Omega \to H$.

Definition 16 (Hierarchical Description)**.** *Given a knowledge space $\overrightarrow{\Omega} = \langle (\Omega, \mathcal{D}), (t_i)_{i \in I}, \Theta \rangle$, we define the (hierarchical) description map $h : \Omega \to H$ as follows.*

1. *For $\alpha = 0$, we define $h^0 : \Omega \to H^0$ by $h^0 := \Theta$.*

2. *For a successor ordinal $\alpha = \beta + 1$, we define $h^\alpha : \Omega \to H^\alpha$ by*

$$h^\alpha(\omega) := (h^\beta(\omega), t(\omega) \circ (h^\beta)^{-1}) := (h^\beta(\omega), (t_i(\omega) \circ (h^\beta)^{-1})_{i \in I}) \text{ for all } \omega \in \Omega.$$

3. *For any limit ordinal α with $(0 <)\alpha \leq \kappa$, we let $h^\alpha : \Omega \to H^\alpha$ be the unique mapping which satisfies $h^\beta = \pi^{\alpha,\beta} \circ h^\alpha$ for all $\beta < \alpha$.*

We omit the superscript κ when $\alpha = \kappa$, i.e., we let $h = h^\kappa$. We call each $h(\omega)$ to be the (hierarchical) description of ω.

In light of Equation (8), for each $\alpha \geq 1$, we can identify each $h^\alpha \in H^\alpha$ in Definition 16 with $h^\alpha = (h^0, (h_i^\alpha)_{i \in I})$ as follows. Let $h_i^1(\omega) := t_i(\omega) \circ (h^0)^{-1}$ for all $\omega \in \Omega$. For a successor ordinal $\alpha = \beta + 1$ with $\beta \geq 1$, we define $h_i^\alpha : \Omega \to H_i^\alpha$ by $h_i^\alpha(\omega) := (h_i^\beta(\omega), t_i(\omega) \circ (h^\beta)^{-1})$ for all $\omega \in \Omega$. For a limit ordinal α, we define $h_i^\alpha : \Omega \to H_i^\alpha$ by the unique mapping which satisfies $h_i^\beta = \pi_i^{\alpha,\beta} \circ h_i^\alpha$ for all $\beta < \alpha$.

Now, we define an underlying set Ω^* of a candidate universal knowledge space as the set of all descriptions of states of the world ranged over all knowledge spaces of I on $(S, \mathcal{A}_S)$. That is,

$$\Omega^* := \{\omega^* \in H \mid \omega^* = h(\omega) \text{ for some } \overrightarrow{\Omega} \text{ and } \omega \in \Omega\}. \tag{9}$$

It is clear that Ω^* is not empty as long as there is a knowledge space $\overrightarrow{\Omega}$ with $\Omega \neq \emptyset$ in the given category of knowledge spaces. As we have already seen in Section 3.1, we indeed have $\Omega^* \neq \emptyset$.

Once Ω^* is defined as a subset of H, we can induce a κ-complete algebra $\mathcal{D}^*$ on Ω^* from $\mathcal{H}$ by

$$\mathcal{D}^* = \{(\pi^\alpha|_{\Omega^*})^{-1}(E^\alpha) \in \mathcal{P}(\Omega^*) \mid E^\alpha \in \mathcal{H}^\alpha \text{ for some } \alpha < \kappa\}, \tag{10}$$

where $\pi^\alpha|_{\Omega^*} : \Omega^* \to H^\alpha$ is the restriction of $\pi^\alpha : H \to H^\alpha$ on Ω^* for each $\alpha < \kappa$. Note that $\mathcal{D}^* = \{E \cap \Omega^* \in \mathcal{P}(\Omega^*) \mid E \in \mathcal{H}\}$, since $(\pi^\alpha|_{\Omega^*})^{-1}(\cdot) = (\pi^\alpha)^{-1}(\cdot) \cap \Omega^*$.

From now on, for any knowledge space $\overrightarrow{\Omega}$, we identify the description map h as $h : \Omega \to \Omega^*$. We denote the description map by $h_{\overrightarrow{\Omega}} : \Omega \to \Omega^*$ when we stress its domain. We establish that the description map $h : (\Omega, \mathcal{D}) \to (\Omega^*, \mathcal{D}^*)$ is κ-measurable and that a knowledge morphism preserves hierarchical descriptions as in Corollary 1. Thus, for any two states $(\omega, \varphi(\omega)) \in \Omega \times \Omega'$ where $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a knowledge morphism, they induce the same hierarchy of knowledge.

Lemma 6 (Preservation of Descriptions through Knowledge Morphism)**.** *Let $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$ be knowledge spaces.*

1. *The description map $h_{\overrightarrow{\Omega}} : (\Omega, \mathcal{D}) \to (\Omega^*, \mathcal{D}^*)$ is κ-measurable.*

2. If $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a knowledge morphism, then $h_{\overrightarrow{\Omega}} = h_{\overrightarrow{\Omega'}} \circ \varphi$.

We make the following remark. As in Section 3.1, we say that a knowledge space $\overrightarrow{\Omega}$ is *non-redundant* if its description map h is injective. If a knowledge space $\overrightarrow{\Omega}$ is non-redundant, then $(\Theta, (t_i)_{i\in I}) : \Omega \to S \times M(\Omega)^I$ is injective. This follows because if $(\Theta, (t_i)_{i\in I})(\omega) = (\Theta, (t_i)_{i\in I})(\omega')$ for some $\omega, \omega' \in \Omega$ then $h(\omega) = h(\omega')$ and thus $\omega = \omega'$.

In the third step, we define the mapping $\Theta^* : \Omega^* \to S$ and players' knowledge-type mappings. We define $\Theta^* : \Omega^* \to S$ by the projection $\Theta^* = \pi^0|_{\Omega^*}$. By construction, $\Theta^* : (\Omega^*, \mathcal{D}^*) \to (S, \mathcal{A}_S)$ is κ-measurable. Moreover, for any knowledge space $\overrightarrow{\Omega}$ and $\omega \in \Omega$, we have

$$\Theta(\omega) = \pi^0(h(\omega)) = \Theta^*(h(\omega)).$$

Hence, Θ^* preserves states of nature. Next, we define players' type mappings as follows.

Lemma 7 (Players' Knowledge in Candidate Universal Space)**.** *For each $i \in I$ and $\omega^* \in \Omega^*$, we define*

$$t_i^*(\omega^*)((\pi^\alpha|_{\Omega^*})^{-1}(E^\alpha)) := (\omega^*)_i^{\alpha+1}(E^\alpha) \text{ for each } \alpha < \kappa \text{ and } E^\alpha \in \mathcal{H}^\alpha, \tag{11}$$

where note that $(\omega^*)_i^{\alpha+1} \in M(H^\alpha, \mathcal{H}^\alpha)$.[71] *Then, we have the following.*

1. *$t_i^* : (\Omega^*, \mathcal{D}^*) \to (M(\Omega^*, \mathcal{D}^*), \mathcal{M}(\Omega^*, \mathcal{D}^*))$ is a well-defined κ-measurable mapping.*
2. *t_i^* inherits all the properties of knowledge imposed in the given category of knowledge spaces.*
3. *Fix a knowledge space $\overrightarrow{\Omega}$. Then, for any $\omega \in \Omega$ and $E^* \in \mathcal{D}^*$,*

$$t_i^*(h(\omega))(E^*) = t_i(\omega)(h^{-1}(E^*)). \tag{12}$$

So far, we have established the following: (i) the space $\overrightarrow{\Omega^*} = \langle (\Omega^*, \mathcal{D}^*), (t_i^*)_{i\in I}, \Theta^* \rangle$ is a legitimate knowledge space with $\Omega^* \neq \emptyset$; and (ii) for any given knowledge space $\overrightarrow{\Omega} = \langle (\Omega, \mathcal{D}), (t_i)_{i\in I}, \Theta \rangle$, the description map $h : \Omega \to \Omega^*$ is a knowledge morphism. In the knowledge space $\overrightarrow{\Omega^*}$, players' knowledge at state ω^* are encoded within the state ω^* through Equation (11).

In the fourth and final step, we show that the description map is a unique knowledge morphism from a given knowledge space into the knowledge space $\overrightarrow{\Omega^*}$. As in Lemma 5 in Section 3.1, we show that the description map from $\overrightarrow{\Omega^*}$ into itself is the identity map.

[71] If we denote $\omega^* = (s, (\mu_i)_{i\in I}) \in \Omega^*$ and $\mu_i = (\mu_i^\alpha)_{0\leq\alpha<\kappa}$ with $\mu_i^\alpha \in M(H^\alpha, \mathcal{H}^\alpha)$, then we have $(\omega^*)_i^{\alpha+1} = \mu_i^\alpha$.

Lemma 8 (Description Map of Candidate Universal Space)**.** *The description map* $h^* : \overrightarrow{\Omega^*} \to \Omega^*$ *is the identity map on* Ω^**.*

Thus, we establish that the knowledge space $\overrightarrow{\Omega^*} = \langle (\Omega^*, \mathcal{D}^*), (t_i^*)_{i\in I}, \Theta^* \rangle$ is universal. Also, the mapping $(\Theta^*, (t_i^*)_{i\in I}) : \Omega^* \to S \times M(\Omega^*, \mathcal{D}^*)^I$ is injective. Thus, the universal knowledge space $\overrightarrow{\Omega^*}$ is in a bijective relation to a subset of $S \times M(\Omega^*, \mathcal{D}^*)^I$ that respects given introspective (as well as Kripke) properties in the following sense.

Theorem 3 (Existence of a Universal Knowledge Space)**.** *The knowledge space* $\overrightarrow{\Omega^*} = \langle (\Omega^*, \mathcal{D}^*), (t_i^*)_{i\in I}, \Theta^* \rangle$ *is universal in each given category of knowledge spaces of* I *on* $(S, \mathcal{A}_S)$*. The mapping* $(\Theta^*, (t_i^*)_{i\in I}) : \Omega^* \to \Omega^{**}$ *is bijective, where*

$$\Omega^{**} := \{(s, (\mu_i)_{i\in I}) \in S \times M(\Omega^*, \mathcal{D}^*)^I \mid \textit{there are a knowledge space } \overrightarrow{\Omega} \textit{ and } \omega \in \Omega \textit{ such that } (s, (\mu_i)_{i\in I}) = (\Theta(\omega), (t_i(\omega) \circ h^{-1})_{i\in I})\}.$$

We note that the discussions in Remark 2 in Section 3.1 apply by replacing the appropriate notations. In the Appendix, we establish the following fact by using category theory (specifically, the theory of coalgebra).

Remark 5 (Property of Universal Space with No Introspection)**.** Suppose that no introspective property nor the Kripke property is imposed in the given class of knowledge spaces. Then, $(\Theta^*, (t_i^*)_{i\in I}) : \Omega^* \to S \times M(\Omega^*, \mathcal{D}^*)^I$ is bijective (indeed, a knowledge isomorphism).

We make the following two remarks. First, a mathematical intuition behind Remark 5 is that a knowledge space $\overrightarrow{\Omega} = \langle (\Omega, \mathcal{D}), (t_i)_{i\in I}, \Theta \rangle$ can be identified as a pair of the κ-complete algebra $(\Omega, \mathcal{D})$ and the mapping $(\Theta, (t_i)_{i\in I}) : (\Omega, \mathcal{D}) \to (F(\Omega), \mathcal{F}(\mathcal{D}))$, where $(F(\Omega), \mathcal{F}(\mathcal{D}))$ is a κ-complete algebra satisfying $F(\Omega) = S \times M(\Omega, \mathcal{D})^I$. In the language of category theory, the pair $\langle (\Omega, \mathcal{D}), (\Theta, (t_i)_{i\in I}) \rangle$ is called a coalgebra over the endofunctor F. Thus, the category of knowledge(-type) spaces is seen as the full subcategory of F-coalgebras such that each t_i satisfies the given introspective and/or Kripke properties, if F is taken to be $S \times M(\Omega, \mathcal{D})^I$.[72] If we do not assume any introspective or the Kripke property, then a knowledge space can be seen as a coalgebra. Now, as is well known in the theory of coalgebras (namely, the Lambek lemma [47]), a terminal (final) coalgebra Ω^* is isomorphic to $F(\Omega^*) = S \times M(\Omega^*)^I$.

Second, we briefly discuss the role of the domain specification. To make the exposition simplest, we impose no assumption on players' knowledge. Suppose that the domain of a knowledge space is always the power set. If such a class of knowledge spaces has a universal knowledge space, then we would have to have a bijection

[72] First, see, for example, Kurz [46], Jacobs and Rutten [44], and Rutten [73] for a general treatment of coalgebras. Second, Moss and Viglizzo [65, 66] take a coalgebraic approach to a universal type space. In the type space approach, introspective properties can be encoded by the requirement that each player's type distribution be defined on the product of states of nature and the other players' types.

between Ω^* and $S \times \mathcal{P}(\mathcal{P}(\Omega^*))^I$, where note that $M(\Omega^*)$ is in a bijective relation to $\mathcal{P}(\mathcal{P}(\Omega^*))$. This is clearly impossible. This argument is closely related to Brandenburger's result on the non-existenece of a (belief-)complete possibility structure (Brandenburger [14]).

6.2 Coherent Hierarchies of Knowledge

In the previous subsection, we have established the existence of a universal knowledge space in terms of hierarchies of knowledge-types. The natural question, then, is to examine how our construction relates to the previous literature on universal type spaces, in which the notion of coherent hierarchies plays an important role.

Below, we define a coherent subset of the hierarchies space H and show that the universal knowledge space $\overrightarrow{\Omega^*}$ (established in Theorem 3) is the largest coherent subset of H. Note that the notion of coherency is defined in the context of the knowledge hierarchies H. We start with the definition of a coherent subset of H.

Definition 17 (Coherent Hierarchies of Knowledge). *1. A subset Ω of H is said to be coherent if $(\Omega, \mathcal{D})$ satisfies the following conditions, where $\mathcal{D}$ is a κ-complete algebra on Ω defined by*

$$\mathcal{D} := \{(\pi^\alpha|_\Omega)^{-1}(E^\alpha) \in \mathcal{P}(\Omega) \mid E^\alpha \in \mathcal{H}^\alpha \text{ for some } \alpha < \kappa\}.$$

Take any $(s, (\mu_i)_{i \in I}) \in \Omega$, where $\mu_i = (\mu_i^\alpha)_{0 \leq \alpha < \kappa}$ with $\mu_i^\alpha \in M(H^\alpha, \mathcal{H}^\alpha)$.

- *(a) For any ordinals (α, β) with $0 \leq \beta \leq \alpha < \kappa$, if $(\pi^\alpha|_\Omega)^{-1}(E^\alpha) = (\pi^\beta|_\Omega)^{-1}(F^\beta)$ for some $E^\alpha \in \mathcal{H}^\alpha$ and $F^\beta \in \mathcal{H}^\beta$, then $\mu_i^\alpha(E^\alpha) = \mu_i^\beta(F^\beta)$ for all $i \in I$.*
- *(b) Suppose that Truth Axiom is imposed on player i. If $\mu_i^\alpha(E^\alpha) = 1$ for some $E^\alpha \in \mathcal{H}^\alpha$, then $(s, (\mu_i)_{i \in I}) \in (\pi^\alpha|_\Omega)^{-1}(E^\alpha)$.*
- *(c) Suppose that Positive Introspection is imposed on player i. If $\mu_i^\alpha(E^\alpha) = 1$ for some $E^\alpha \in \mathcal{H}^\alpha$, then $\mu_i^\alpha(\{(s', (\mu'_j)_{j \in I}) \in H \mid (\mu'_i)^\alpha \in \mathcal{M}(E^\alpha)\}) = 1$.*
- *(d) Suppose that Negative Introspection is imposed on player i. If $\mu_i^\alpha(E^\alpha) = 0$ for some $E^\alpha \in \mathcal{H}^\alpha$, then $\mu_i^\alpha(\{(s', (\mu'_j)_{j \in I}) \in H \mid (\mu'_i)^\alpha \in \neg\mathcal{M}(E^\alpha)\}) = 1$.*
- *(e) Suppose that the Kripke property is imposed on player i. If $\bigcap\{E \in \mathcal{D} \mid E = (\pi^\alpha|_\Omega)^{-1}(E^\alpha) \text{ and } \mu_i^\alpha(E^\alpha) = 1 \text{ for some } \alpha < \kappa \text{ and } E^\alpha \in \mathcal{H}^\alpha\} \subseteq (\pi^\beta|_\Omega)^{-1}(F^\beta)$ for some $\beta < \kappa$ and $F^\beta \in \mathcal{H}^\beta$, then $\mu_i^\beta(F^\beta) = 1$.*

2. If Ω is a coherent subset of H, then we define the induced knowledge space $\overrightarrow{\Omega} = \langle(\Omega, \mathcal{D}), (t_i)_{i \in I}, \Theta\rangle$ as follows.

- *(a) We define $\Theta : (\Omega, \mathcal{D}) \to (S, \mathcal{A}_S)$ by the projection $\pi^0|_\Omega$.*
- *(b) We define $t_i : (\Omega, \mathcal{D}) \to (M(\Omega, \mathcal{D}), \mathcal{M}(\Omega, \mathcal{D}))$ by*

$$t_i(s, (\mu_j)_{j \in I})((\pi^\alpha|_\Omega)^{-1}(E^\alpha)) := \mu_i^\alpha(E^\alpha) \text{ for each } E^\alpha \in \mathcal{H}^\alpha.$$

For any subset Ω of H, we can induce a κ-complete algebra $\mathcal{D}$ from $\mathcal{H}$. Also, the mapping Θ is naturally defined by the projection through Condition (2a). Now, in order for Ω to be a coherent subset of H, Condition (1a) requires the set Ω to induce well-defined knowledge-type mappings within Ω through Condition (2b).[73] This condition requires different levels of players' knowledge not to contradict with each other. The other conditions stipulated in (1) ensures that the knowledge-type mappings respect the required properties of knowledge.

Now, we establish our main result of this subsection that $\overrightarrow{\Omega^*}$ is a largest coherent subset of H.

Theorem 4 (Universal Knowledge Space is Largest Coherent Space). *The universal knowledge space Ω^* established in Section 6.1 is the largest coherent subset of H in the following sense: (i) Ω^* is coherent; and (ii) for any coherent subset Ω of H, the description map $h : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ is an inclusion map (so that $\Omega \subseteq \Omega^*$).*

7 Applications to Richer Settings

In this section, we discuss applicability of our framework. Specifically, since our framework can accommodate a case where players have multiple epistemic operators, we can consider the following extensions. First, we can add time dynamics so that each player's knowledge at each time is described by her knowledge operator at that time. Second, we can add other epistemic operators such as non-probabilistic belief and unawareness.[74]

In Section 7.1, we consider dynamic knowledge-belief spaces, where players have qualitative beliefs as well as knowledge in a dynamic setting. Next, in Section 7.2, we consider knowledge-unawareness spaces in the context of state space models. Henceforth in this section, fix a κ-complete algebra of states of nature $(S, \mathcal{A}_S)$ and a non-empty set of players I, where κ is an infinite regular cardinal.

7.1 Universal Dynamic Knowlwdge-(Non-probabilistic-)Belief Spaces

Epistemic analyses of dynamic games usually call for players' knowledge and belief. We demonstrate that there exists a universal dynamic knowledge-belief space, where players' beliefs are defined by their qualitative (i.e., non-probabilistic) belief operators. The class of dynamic knowledge-belief spaces that we study is based on Battigali and Bonanno [9].

[73] The corresponding condition that induces players' knowledge in a well-defined manner in the syntactic approach is Condition (2a) in Theorem 2 of Section 3.2.

[74] Although we entirely omit it from our discussion, counterfactual/hypothetical reasoning would be another interesting example.

We assume, for simplicity, that time runs through $\mathbb{N}$. Each player i's knowledge at time $t \in \mathbb{N}$ is represented by a knowledge operator $K_{i,t} : \mathcal{D} \to \mathcal{D}$, while player i's (non-probabilistic) belief at time $t \in \mathbb{N}$ is captured by a belief operator on the same domain: $B_{i,t} : \mathcal{D} \to \mathcal{D}$. Specifically, we define a dynamic knowledge-belief space of I on $(S, \mathcal{A}_S)$ as follows.

Definition 18 (Dynamic Knowledge-Belief Space). *A dynamic (κ-)knowledge-belief space of I on $(S, \mathcal{A}_S)$ is a tuple $\overrightarrow{\Omega} = \langle (\Omega, \mathcal{D}), (K_{i,t})_{(i,t) \in I \times \mathbb{N}}, (B_{i,t})_{(i,t) \in I \times \mathbb{N}}, \Theta \rangle$ with the following properties:*

1. *$(\Omega, \mathcal{D})$ is a κ-complete algebra and the mapping $\Theta : (\Omega, \mathcal{D}) \to (S, \mathcal{A}_S)$ is κ-measurable.*
2. *Each knowledge operator $K_{i,t} : \mathcal{D} \to \mathcal{D}$ satisfies Truth Axiom, Positive Introspection, Monotonicity, κ-Conjunction, Necessitation, and Negative Introspection.*
3. *Each belief operator $B_{i,t} : \mathcal{D} \to \mathcal{D}$ satisfies Consistency, Positive Introspection, Monotonicity, κ-Conjunction, Necessitation, and Negative Introspection.*
4. *Knowledge and belief operators satisfy the following joint conditions.*
   - *(a) $K_{i,t}(E) \subseteq B_{i,t}(E)$ for any $E \in \mathcal{D}$.*
   - *(b) $B_{i,t}(E) \subseteq K_{i,t}B_{i,t}(E)$ for any $E \in \mathcal{D}$.*
   - *(c) $B_{i,t}(E) = B_{i,t}B_{i,t+1}(E)$ for each $E \in \mathcal{D}$.*

Conditions (4) specify relations between knowledge and belief. The first condition (4a) means that knowledge implies belief at each time. The second (4b) states that each player knows her own belief at each time, where note that $(\neg B_{i,t})(\cdot) \subseteq K_{i,t}(\neg B_{i,t})(\cdot)$ is satisfied, given Truth Axiom and Negative Introspection of knowledge. The third condition (4c) captures the idea of belief persistence by Battigali and Bonanno [9]: player i believes E at times t iff she believes at t that she (will) believe E at $t+1$. We say that player i's knowledge satisfies perfect recall if $K_{i,t}(E) \subseteq K_{i,t+1}(E)$ for all $E \in \mathcal{D}$ and $t \in \mathbb{N}$. A dynamic knowledge-belief space *with perfect recall* is a dynamic knowledge-belief space such that each player's knowledge satisfies perfect recall.

The point is that a dynamic knowledge-belief space is mathematically a knowledge space of $I \times \mathbb{N} \times \{0, 1\}$, where "player" $(i, t, 0)$'s knowledge operator is i's knowledge operator at time t while "player" $(i, t, 1)$'s knowledge operator is i's (non-probabilistic) belief operator at time t, with the specified conditions.

A knowledge-belief morphism can be regarded as a knowledge morphism of this extended knowledge space. That is, a (dynamic) knowledge-belief morphism from $\overrightarrow{\Omega}$ to $\overrightarrow{\Omega'}$ is a κ-measurable mapping $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ with the following properties:

(i) $\Theta' \circ \varphi = \Theta$; and (ii) for all $i \in I$ and $t \in \mathbb{N}$, $K_{i,t}(\varphi^{-1}(\cdot)) = \varphi^{-1}(K'_{i,t}(\cdot))$ and $B_{i,t}(\varphi^{-1}(\cdot)) = \varphi^{-1}(B'_{i,t}(\cdot))$.

A universal dynamic knowledge-belief space is a universal knowledge space in this category of extended knowledge spaces. That is, a dynamic knowledge-belief space $\overrightarrow{\Omega^*}$ of I on $(S, \mathcal{A}_S)$ is said to be *universal* if, for any dynamic knowledge-belief space Ω of I on $(S, \mathcal{A}_S)$ there is a unique (dynamic) knowledge-belief morphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$.

Our previous arguments apply to the existence of a universal dynamic knowledge-belief space.

Theorem 5 (Existence of Universal Dynamic Knowledge-Belief Space)**.** *There exists a universal dynamic knowledge-belief space (with/without perfect recall) $\overrightarrow{\Omega^*}$ of I on $(S, \mathcal{A}_S)$.*

7.2 Universal Knowledge-Unawareness Spaces

In our framework, we can accommodate a wide variety of assumptions on players' knowledge. This rises questions as to how our framework can accommodate notions of unawareness. While studies of unawareness in the framework goes beyond the scope of our present paper, we illustrate the idea that our arguments can be applied to the existence of a universal space involving a notion of unawareness. We do so within the (simple) framework of standard state space models (see, for example, Chen, Ely, and Luo [19] and DLR [21]).[75]

Specifically, we represent each player i's knowledge and unawareness by her knowledge operator $K_i : \mathcal{D} \to \mathcal{D}$ and unawareness operator $U_i : \mathcal{D} \to \mathcal{D}$, respectively, where $\mathcal{D}$ is a κ-complete algebra on an underlying set Ω. Player i is *unaware* of an event $E \in \mathcal{D}$ at a state $\omega \in \Omega$ if $\omega \in U_i(E)$. On the other hand, player i is *aware* of an event $E \in \mathcal{D}$ at a state $\omega \in \Omega$ if $\omega \in A_i(E) := (\neg U_i)(E)$.

It is to be noted that any set $E \in \mathcal{P}(\Omega) \setminus \mathcal{D}$ is simply not an object of players' unawareness. Hence, the fact that a player is unaware of a certain event should be distinguished from the fact that some event is not an object of her unawareness.

While we can establish the existence of a universal knowledge-unawareness space where the notion of unawareness satisfies various other properties, here we restrict attention to the following three axioms on players' unawareness taken from DLR [21].[76]

The first axiom is Plausibility: $U_i(\cdot) \subseteq (\neg K_i)(\cdot) \cap (\neg K_i)^2(\cdot)$. It states that if a player is unaware of an event E then she does not know E and she does not know

[75] First, as Modica and Rustichini [62] and DLR [21] demonstrate, standard state space models have limitations in representing unawareness. Second, it would be an interesting question to ask whether or how our results in this paper apply to models of unawareness that have richer structures than standard state space models (see, Schipper [79] and the references therein for the richer structures). Third, a companion paper (Fukuda [30]) studies how a standard state space model can (and cannot) represent notions of unawareness within our present framework.

[76] For other properties of unawareness, see Schipper [79] and the references therein.

that she does not know E. The second is KU Introspection: $K_i U_i(\cdot) = \emptyset$. It requires that any player cannot know any event of which she is unaware. The third is AU Introspection: $U_i(\cdot) \subseteq U_i U_i(\cdot)$. It says that if a player is unaware of an event then she is unaware of being unaware of that event.

Definition 19 (Knowledge-Unawareness Space). *A knowledge-unawareness space of I on $(S, \mathcal{A}_S)$ is a tuple $\overrightarrow{\Omega} := \langle (\Omega, \mathcal{D}), (K_i)_{i \in I}, (U_i)_{i \in I}, \Theta \rangle$ with the following properties.*

1. *$\langle (\Omega, \mathcal{D}), (K_i)_{i \in I}, \Theta \rangle$ is a κ-knowledge space of I on $(S, \mathcal{A}_S)$.*
2. *Each unawareness operator $U_i : \mathcal{D} \to \mathcal{D}$ satisfies (some of) Plausibility, KU Introspection, and AU Introspection.*

We say that a knowledge-unawareness space $\overrightarrow{\Omega}$ captures a non-trivial form of unawareness ($\overrightarrow{\Omega}$ is non-trivial, for short) if there is $(i, E) \in I \times \mathcal{D}$ such that $U_i(E) \neq \emptyset$.

Note that we can regard a knowledge-unawareness space as a knowledge space by identifying unawareness operators with "knowledge" operators. With keeping this in mind, we define a knowledge-unawareness morphism and a universal knowledge-unawareness space. Also, note that a given knowledge space induces a knowledge-unawareness space, where unawareness operators are, for example, defined by $U_i(\cdot) = (\neg K_i)(\cdot) \cap (\neg K_i)^2(\cdot)$.

A κ-measurable mapping $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ between knowledge-unawareness spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$ is called a knowledge-unawareness morphism if (i) $\Theta' \circ \varphi = \Theta$ and if (ii) $K_i(\varphi^{-1}(\cdot)) = \varphi^{-1}(K'_i(\cdot))$ and $U_i(\varphi^{-1}(\cdot)) = \varphi^{-1}(U'_i(\cdot))$ for all $i \in I$.

We say that a knowledge-unawareness space $\overrightarrow{\Omega^*}$ of I on $(S, \mathcal{A}_S)$ is *universal* if, for any knowledge-unawareness space Ω of I on $(S, \mathcal{A}_S)$ there is a unique knowledge-unawareness morphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$.

Now, we show that there exists a universal knowledge-unawareness space Ω^* of I on $(S, \mathcal{A}_S)$. We also present a mild sufficient condition for guaranteeing its non-triviality.[77]

Theorem 6 (Existence of Universal Knowledge-Unawareness Space). *There exists a universal knowledge-unawareness space $\overrightarrow{\Omega^*}$ of I on $(S, \mathcal{A}_S)$. Moreover, suppose that there is a knowledge-unawareness space $\overrightarrow{\Omega}$ in a given category of knowledge-unawareness spaces such that $U_i([\![e]\!]_{\overrightarrow{\Omega}}) \neq \emptyset$ for some $i \in I$ and $e \in \mathcal{L}$. Then, $\overrightarrow{\Omega^*}$ is non-trivial.*

[77] In light of this result, here we do not study conditions on the knowledge and unawareness operators which guarantee a non-trivial form of unawareness. A companion paper (Fukuda [30]) studies this question.

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A Appendix

A.1 Section 2

Remark A.1 (Proof of Equation (1))**.** Since $\omega' \notin \{\omega'\}^c$, if $\omega' \in b_{K_i}(\omega)$ then $\omega \in (\neg K_i)(\{\omega'\}^c) = L_{K_i}(\{\omega'\})$. Conversely, suppose to the contrary that there is $E \in \mathcal{D}$ with $\omega' \notin E$ (i.e., $E \subseteq \{\omega'\}^c$) and $\omega \in K_i(E)$. By Monotonicity, we have $\omega \in K_i(\{\omega'\}^c)$, a contradiction.

Proposition A.1 (Criteria for the Kripke Property)**.** *Let $\overrightarrow{\Omega}$ be a knowledge space such that K_i satisfies Monotonicity. The Kripke property of K_i is characterized as follows. If $(\omega, F) \in \Omega \times \mathcal{D}$ satisfies that $E \cap F \neq \emptyset$ for all $E \in \mathcal{D}$ with $\omega \in K_i(E)$, then (ω, F) satisfies $b_{K_i}(\omega) \cap F \neq \emptyset$.*

Proof of Proposition A.1. Assume the Kripke property. Take any $(\omega, F) \in \Omega \times \mathcal{D}$ such that $E \cap F \neq \emptyset$ for all $E \in \mathcal{D}$ with $\omega \in K_i(E)$. Now, if it were the case that $b_{K_i}(\omega) \cap F = \emptyset$, then we would have $b_{K_i}(\omega) \subseteq F^c$ and thus $F^c \cap F \neq \emptyset$, a contradiction.

Conversely, suppose to the contrary that there is $E' \in \mathcal{D}$ such that $b_{K_i}(\omega) \subseteq E'$ and $\omega \notin K_i(E')$. Since $b_{K_i}(\omega) \cap (E')^c = \emptyset$, there is $E \in \mathcal{D}$ such that $\omega \in K_i(E)$ and $E \cap (E')^c = \emptyset$. Then, we have $E \subseteq E'$. By Monotonicity, $\omega \in K_i(E')$, which is a contradiction. $\square$

Remark A.2 (Class of Knowlede Spaces as a Category)**.** First, the category of knowledge spaces is large (but locally small), whenever $S \neq \emptyset$. Indeed, we can introduce players' knowledge for any set Ω. Second, let $\mathbf{K}$ and $\mathbf{K}'$ be two categories of knowledge spaces where the assumptions on players' knowledge in $\mathbf{K}'$ are also assumed in $\mathbf{K}$. Then, $\mathbf{K}$ is a full subcategory of $\mathbf{K}'$: (i) any $\mathbf{K}$-object is also a $\mathbf{K}'$-object; (ii) any $\mathbf{K}$-morphism is also a $\mathbf{K}'$-morphism with the same identity and composite morphisms; and (iii) for any $\mathbf{K}$-objects $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$ and for any $\mathbf{K}'$-morphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$, the mapping φ is also a $\mathbf{K}$-morphism.

A.2 Section 3

A.2.1 Section 3.1

First, Figure 1 illustrates the interrelations among the definitions and lemmas for the construction of a universal knowledge space in established Theorem 1.

Proof of Remark 1. We first show that $\mathcal{L}_\kappa \subseteq \mathcal{L}$. Clearly, $\mathcal{L}_0 \subseteq \mathcal{L}_\kappa$. If $\mathcal{L}_\beta \subseteq \mathcal{L}$ for all $\beta < \alpha$, then it is clear that $\mathcal{L}_\alpha \subseteq \mathcal{L}$. Hence, $\mathcal{L}_\kappa \subseteq \mathcal{L}$.

Conversely, it can be seen that if $e \in \mathcal{L}_\kappa$ then there is $\alpha < \kappa$ such that $e \in \mathcal{L}_\alpha$. Now, we establish that $\mathcal{L} \subseteq \mathcal{L}_\kappa$. First, it is immediate that $\mathcal{A}_S \subseteq \mathcal{L}_\kappa$. Second, if $e \in \mathcal{L}_\kappa$ then $e \in \mathcal{L}_\alpha$ for some $\alpha < \kappa$ and thus $(\neg e) \in \mathcal{L}_{\alpha+1} \subseteq \mathcal{L}_\kappa$. Third, let $\mathcal{F}$ be such that $\mathcal{F} \subseteq \mathcal{L}_\kappa$ and $0 < |\mathcal{F}| < \kappa$. Then, there is $\gamma < \kappa$ such that $e \in \mathcal{L}_\gamma \subseteq \mathcal{L}_\kappa$ for all $e \in \mathcal{F}$, and thus $\bigwedge \mathcal{F} \in \mathcal{L}_\kappa$. Hence, $\mathcal{L} \subseteq \mathcal{L}_\kappa$. $\square$

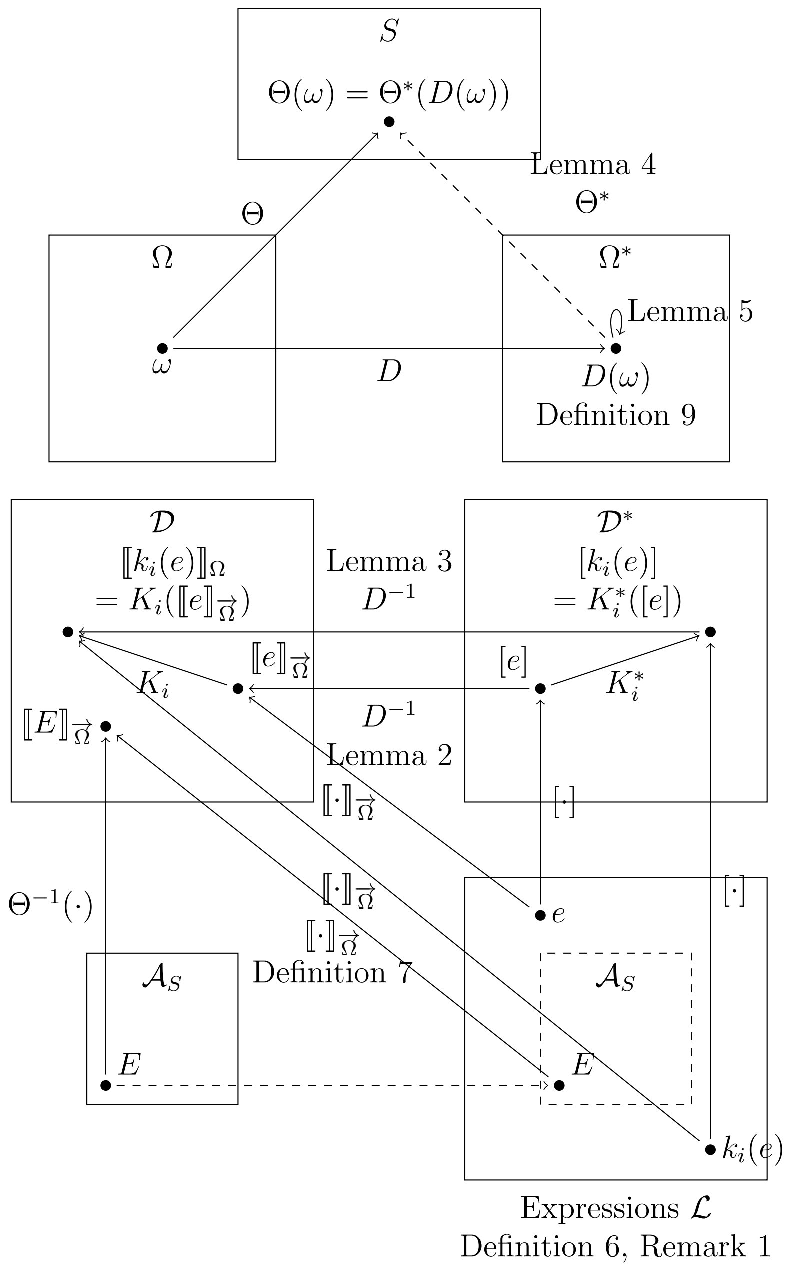

Figure 1: Interrelations among the Definitions and Lemmas for Theorem 1.

Proof of Lemma 1. In a similar way to the proof of HS [39, Proposition 4.1], Meier [55, Proposition 2], and Meier [56, Proposition 1], we show the statement by induction on the formulation of the expressions.

First, for each $E \in \mathcal{A}_S$, we have: $\omega \in [\![E]\!]_{\overrightarrow{\Omega}} := \Theta^{-1}(E)$ iff $\Theta(\omega) \in E$ iff $\Theta'(\varphi(\omega)) \in E$ iff $\varphi(\omega) \in (\Theta')^{-1}(E) =: [\![E]\!]_{\overrightarrow{\Omega'}}$, where $\Theta(\omega) = \Theta'(\varphi(\omega))$ follows from the assumption that φ is a knowledge morphism.

Second, let $\mathcal{E}$ be a set of expressions with $|\mathcal{E}| < \kappa$. If $\mathcal{E} = \emptyset$, then observe that $\bigwedge \emptyset := S \in \mathcal{A}_S$. Hence, we let $\mathcal{E} \neq \emptyset$. Given the induction hypothesis, we have

$$\omega \in \left[\!\left[\bigwedge \mathcal{E} \right]\!\right]_{\overrightarrow{\Omega}} := \bigcap_{e \in \mathcal{E}} [\![e]\!]_{\overrightarrow{\Omega}} \text{ iff } \omega \in [\![e]\!]_{\overrightarrow{\Omega}} \text{ for all } e \in \mathcal{E}$$
$$\text{iff } \varphi(\omega) \in [\![e]\!]_{\overrightarrow{\Omega'}} \text{ for all } e \in \mathcal{E} \text{ iff } \varphi(\omega) \in \bigcap_{e \in \mathcal{E}} [\![e]\!]_{\overrightarrow{\Omega'}} =: \left[\!\left[\bigwedge \mathcal{E} \right]\!\right]_{\overrightarrow{\Omega'}}.$$

Third, assuming the induction hypothesis, we get

$$\omega \in [\![\neg e]\!]_{\overrightarrow{\Omega}} := \neg [\![e]\!]_{\overrightarrow{\Omega}} \text{ iff } \omega \notin [\![e]\!]_{\overrightarrow{\Omega}} \text{ iff } \varphi(\omega) \notin [\![e]\!]_{\overrightarrow{\Omega'}} \text{ iff } \varphi(\omega) \in \neg [\![e]\!]_{\overrightarrow{\Omega'}} =: [\![\neg e]\!]_{\overrightarrow{\Omega'}}.$$

Fourth, assuming the induction hypothesis, we have

$$\omega \in [\![k_i(e)]\!]_{\overrightarrow{\Omega}} := K_i([\![e]\!]_{\overrightarrow{\Omega}}) = K_i(\varphi^{-1}([\![e]\!]_{\overrightarrow{\Omega'}})) = \varphi^{-1}(K'_i([\![e]\!]_{\overrightarrow{\Omega'}}))$$
$$\text{iff } \varphi(\omega) \in K'_i([\![e]\!]_{\overrightarrow{\Omega'}}) =: [\![k_i(e)]\!]_{\overrightarrow{\Omega'}}.$$

The induction is complete. □

Proof of Corollary 1. Fix $\omega \in \Omega$. First, since φ is a knowledge morphism, we have $\Theta(\omega) = \Theta'(\varphi(\omega))$. Second, it follows from Lemma 1 that $\omega \in [\![e]\!]_{\overrightarrow{\Omega}}$ iff $\varphi(\omega) \in [\![e]\!]_{\overrightarrow{\Omega'}}$ for all $e \in \mathcal{L}$. It follows from these two arguments that $D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega'}}(\varphi(\omega))$. □

Remark A.3 (Remark on Footnote 28)**.** Two states, which possibly reside in different knowledge spaces, are identified when the descriptions are identical. We remark that this notion is related to behavioral equivalence (Kurz [46]). Let $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$ be knowledge spaces in a given category. A pair of states $(\omega, \omega') \in \Omega \times \Omega'$ is said to be *behaviorally equivalent* if there are a knowledge space $\overrightarrow{\Omega''}$ and a pair of knowledge morphisms $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega''}$ and $\varphi' : \overrightarrow{\Omega'} \to \overrightarrow{\Omega''}$ such that $\varphi(\omega) = \varphi'(\omega')$.

We show that $(\omega, \omega') \in \Omega \times \Omega'$ is behaviorally equivalent iff $D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega'}}(\omega')$. This means that, in order to show that two states are identical in terms of players' hierarchies of knowledge, it is enough to show that they are behaviorally equivalent. The proof goes as follows. If $(\omega, \omega') \in \Omega \times \Omega'$ is behaviorally equivalent then we have

$$D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega''}}(\varphi(\omega)) = D_{\overrightarrow{\Omega''}}(\varphi'(\omega')) = D_{\overrightarrow{\Omega'}}(\omega'),$$

where the first and third equalities follow from Corollary 1. The converse trivially holds once we show that the description map is a knowledge morphism.

Proof of Lemma 2. We divide the proof into the following three steps. In the first step, we prove the following correspondence between syntactic and semantic operations.

1. $[(\neg e)] = \neg [e]$ for any $e \in \mathcal{L}$.

2. $[S] = \Omega^*$ and $[\emptyset] = \emptyset$. In other words, $[\bigwedge \emptyset] = \Omega^*$ and $[\bigvee \emptyset] = \emptyset$.

3. Let $\mathcal{E}$ be a set of expressions with $(0 <)|\mathcal{E}| < \kappa$. Then, $\left[\bigwedge_{e\in\mathcal{E}} e\right] = \bigcap_{e\in\mathcal{E}}[e]$ and $\left[\bigvee_{e\in\mathcal{E}} e\right] = \bigcup_{e\in\mathcal{E}}[e]$.

1. Fix $e \in \mathcal{L}$. We have the following:

$$\omega^* \in [(\neg e)] \text{ iff } (1, (\neg e)) \in \omega^* = D(\omega) \text{ iff } \omega \in [\![\neg e]\!]_{\overrightarrow{\Omega}} = \neg [\![e]\!]_{\overrightarrow{\Omega}}$$
$$\text{iff } \omega \notin [\![e]\!]_{\overrightarrow{\Omega}} \text{ iff } (1, e) \notin D(\omega) = \omega^* \text{ iff } \omega^* \notin [e] \text{ iff } \omega^* \in (\neg[e]),$$

where a knowledge space $\overrightarrow{\Omega}$ and a state $\omega \in \Omega$ satisfy $\omega^* = D(\omega)$. Thus, we obtain $[(\neg e)] = \neg[e]$.

2. First, we show that $[S] = \Omega^*$. It is sufficient to prove that $\Omega^* \subseteq [S]$. For any $\omega^* \in \Omega^*$, there are a knowledge space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $\omega^* = D(\omega)$. Now, noting that $[\![S]\!]_{\overrightarrow{\Omega}} = \Theta^{-1}(S) = \Omega$, we have $\omega \in \Omega = [\![S]\!]_{\overrightarrow{\Omega}}$. By definition, we get $(1, S) \in D(\omega) = \omega^*$ and thus $\omega^* \in [S]$. Hence, $\Omega^* \subseteq [S]$. Now, we also have $[\emptyset] = [\neg S] = \neg[S] = \emptyset$.

3. Let $\mathcal{E}$ be a set of expressions with $(0 <)|\mathcal{E}| < \kappa$. It is enough to show that $[\bigwedge \mathcal{E}] = \bigcap_{e\in\mathcal{E}}[e]$. If $\omega^* \in [\bigwedge \mathcal{E}]$, then $(1, \bigwedge \mathcal{E}) \in \omega^*$. There are a knowledge space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $D(\omega) = \omega^*$. Thus, we have $(1, \bigwedge \mathcal{E}) \in \omega^* = D(\omega)$. By Definitions 7 and 9, we have $\omega \in [\![\bigwedge \mathcal{E}]\!]_{\overrightarrow{\Omega}} = \bigcap_{e\in\mathcal{E}} [\![e]\!]_{\overrightarrow{\Omega}}$. Since $\omega \in [\![e]\!]_{\overrightarrow{\Omega}}$ iff $(1, e) \in D(\omega) = \omega^*$ iff $\omega^* \in [e]$, it follows that $\omega^* \in \bigcap_{e\in\mathcal{E}}[e]$.

   Conversely, suppose $\omega^* \in \bigcap_{e\in\mathcal{E}}[e]$. There are a knowledge space $\overrightarrow{\Omega'}$ and ω' such that $D(\omega') = \omega^*$. Again, since $\omega^* \in [e]$ iff $(1, e) \in D(\omega') = \omega^*$ iff $\omega' \in [\![e]\!]_{\overrightarrow{\Omega'}}$, we have $\omega' \in \bigcap_{e\in\mathcal{E}} [\![e]\!]_{\overrightarrow{\Omega'}} = [\![\bigwedge \mathcal{E}]\!]_{\overrightarrow{\Omega'}}$. It follows from Definition 9 that $(1, \bigwedge \mathcal{E}) \in D(\omega') = \omega^*$, i.e., $\omega^* \in [\bigwedge \mathcal{E}]$.

In the second step, we show that $\mathcal{D}^*$ is a κ-complete algebra on Ω^*. It follows from the first step that $\emptyset = [\emptyset] \in \mathcal{D}^*$ and $\Omega^* = [S] \in \mathcal{D}^*$, where note that $\emptyset$ and S are expressions. In other words, $\mathcal{D}^*$ is closed under the empty intersection and union.

Next, we show that $\mathcal{D}^*$ is closed under complementation. If $[e] \in \mathcal{D}^*$, then it follows from the first step that $\neg[e] = [(\neg e)]$. Since $(\neg e) \in \mathcal{L}$, we have $\neg[e] = [(\neg e)] \in \mathcal{D}^*$.

Next, we show that $\mathcal{D}^*$ is closed under non-empty κ-intersection (and κ-union). It is enough to take a subset $\mathcal{E}$ of $\mathcal{L}$ with $(0 <)|\mathcal{E}| < \kappa$. Then, it follows from the first step that $\bigcap_{e\in\mathcal{E}}[e] = [\bigwedge \mathcal{E}]$ (and $\bigcup_{e\in\mathcal{E}}[e] = [\bigvee \mathcal{E}]$). Since $\bigwedge \mathcal{E} \in \mathcal{L}$ (and $\bigvee \mathcal{E} \in \mathcal{L}$), we have $\bigcap_{e\in\mathcal{E}}[e] = [\bigwedge \mathcal{E}] \in \mathcal{D}^*$ (and $\bigcup_{e\in\mathcal{E}}[e] = [\bigvee \mathcal{E}] \in \mathcal{D}^*$).

In the third step, we show that, for any given knowledge space $\overrightarrow{\Omega}$, the description map $D : \Omega \to \Omega^*$ satisfies that $D^{-1}([e]) = [\![e]\!]_{\overrightarrow{\Omega}}$ for all $[e] \in \mathcal{D}^*$. Fix $[e] \in \mathcal{D}^*$. We have $\omega \in D^{-1}([e])$ iff $D(\omega) \in [e]$ iff $(1, e) \in D(\omega)$ iff $\omega \in [\![e]\!]_{\overrightarrow{\Omega}}$. $\square$

Next, we establish Lemma 3 in a general way so that we can also demonstrate the preservation of other potential properties on knowledge. To that end, we consider the following two lemmas.

Lemma A.1 (Preservation of Varieties of Properties of Knowledge)**.** *We express a property of players' knowledge by using operators $f_{\overrightarrow{\Omega}} : \mathcal{D} \to \mathcal{D}$ and $g_{\overrightarrow{\Omega}} : \mathcal{D} \to \mathcal{D}$ for each κ-knowledge space $\overrightarrow{\Omega}$. To that end, suppose that, for any given pair of knowledge spaces $(\overrightarrow{\Omega}, \overrightarrow{\Omega'})$ and knowledge morphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$, the operations $f_{\overrightarrow{\Omega}}$ and $f_{\overrightarrow{\Omega'}}$ (likewise, $g_{\overrightarrow{\Omega}}$ and $g_{\overrightarrow{\Omega'}}$) satisfy $\varphi^{-1}(f_{\overrightarrow{\Omega'}}(E')) = f_{\overrightarrow{\Omega}}(\varphi^{-1}(E'))$ (likewise, $\varphi^{-1}(g_{\overrightarrow{\Omega'}}(E')) = g_{\overrightarrow{\Omega}}(\varphi^{-1}(E'))$) for all $E' \in \mathcal{D}'$.*

For example, $f_{\overrightarrow{\Omega}}$ and $g_{\overrightarrow{\Omega}}$ are operators generated by composing knowledge operators $(K_i)_{i \in I}$ and set-algebraic as well as constant and identity operations. Specific examples are: $f_{\overrightarrow{\Omega}}(\cdot) = K_i(\cdot)$, $f_{\overrightarrow{\Omega}}(\cdot) = K_i(\neg K_i)(\cdot)$, $f_{\overrightarrow{\Omega}}(\cdot) = \bigcap_{i \in I} K_i(\cdot)$, $f_{\overrightarrow{\Omega}}(\cdot) = \mathrm{id}_{\mathcal{D}}(\cdot)$, $f_{\overrightarrow{\Omega}}(\cdot) = K_i K_j(\cdot)$, $f_{\overrightarrow{\Omega}}(\cdot) = K_i(\emptyset)$, $f_{\overrightarrow{\Omega}}(\cdot) = \Omega$, $f_{\overrightarrow{\Omega}}(\cdot) = \emptyset$, and so on.

Now, we have the following.

1. *Suppose that $\omega \in g_{\overrightarrow{\Omega}}(E)$ for all $E \in \mathcal{D}$ with $\omega \in f_{\overrightarrow{\Omega}}(E)$. Then, for any $\omega \in \Omega$ and $E' \in \mathcal{D}$, if $\varphi(\omega) \in f_{\overrightarrow{\Omega}}(E')$ then $\varphi(\omega) \in g_{\overrightarrow{\Omega}}(E')$.*

2. *(a) If $f_{\overrightarrow{\Omega}}(\cdot) \subseteq g_{\overrightarrow{\Omega}}(\cdot)$ then $f_{\overrightarrow{\Omega^*}}(\cdot) \subseteq g_{\overrightarrow{\Omega^*}}(\cdot)$.*[A.1]

   (b) Suppose that $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a surjective knowledge morphism. If $f_{\overrightarrow{\Omega}}(\cdot) \subseteq g_{\overrightarrow{\Omega}}(\cdot)$ then $f_{\overrightarrow{\Omega'}}(\cdot) \subseteq g_{\overrightarrow{\Omega'}}(\cdot)$.

3. *Suppose that $f_{\overrightarrow{\Omega}}(E) \subseteq f_{\overrightarrow{\Omega}}(F)$ for all $E, F \in \mathcal{D}$ with $E \subseteq F$. Then, $\varphi(\omega) \in f_{\overrightarrow{\Omega'}}(E')$ implies $\varphi(\omega) \in f_{\overrightarrow{\Omega'}}(F')$ for all $E', F' \in \mathcal{D}'$ with $E' \subseteq F'$ and $\omega \in \Omega$.*

4. *Suppose that $f_{\overrightarrow{\Omega}}(E) \subseteq f_{\overrightarrow{\Omega}}(F)$ for all $E, F \in \mathcal{D}$ with $E \subseteq F$.*

   (a) $f_{\overrightarrow{\Omega^*}}([e]) \subseteq f_{\overrightarrow{\Omega^*}}([f])$ for all $[e], [f] \in \mathcal{D}^*$ with $[e] \subseteq [f]$.

   (b) Suppose that $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a surjective knowledge morphism. Then, $f_{\overrightarrow{\Omega'}}(E') \subseteq f_{\overrightarrow{\Omega'}}(F')$ for all $E', F' \in \mathcal{D}'$ with $E' \subseteq F'$.

5. *Suppose that $\bigcap_{E \in \mathcal{E}} f_{\overrightarrow{\Omega}}(E) \subseteq f_{\overrightarrow{\Omega}}(\bigcap \mathcal{E})$ for all $\mathcal{E} \subseteq \mathcal{D}$ with $(0 <)|\mathcal{E}| < \kappa$. Then, $\varphi(\omega) \in \bigcap_{E' \in \mathcal{E}'} f_{\overrightarrow{\Omega'}}(E')$ implies $\varphi(\omega) \in f_{\overrightarrow{\Omega'}}(\bigcap \mathcal{E}')$ for all $\omega \in \Omega$ and $\mathcal{E}' \subseteq \mathcal{D}'$ with $(0 <)|\mathcal{E}'| < \kappa$.*

6. *Suppose that $\bigcap_{E \in \mathcal{E}} f_{\overrightarrow{\Omega}}(E) \subseteq f_{\overrightarrow{\Omega}}(\bigcap \mathcal{E})$ for all $\mathcal{E} \subseteq \mathcal{D}$ with $(0 <)|\mathcal{E}| < \kappa$.*

   (a) $\bigcap_{[e] \in \mathcal{E}^*} f_{\overrightarrow{\Omega^*}}([e]) \subseteq f_{\overrightarrow{\Omega^*}}(\bigcap \mathcal{E}^*)$ for all $\mathcal{E}^* \subseteq \mathcal{D}^*$ with $(0 <)|\mathcal{E}^*| < \kappa$.

   (b) Suppose that $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a surjective knowledge morphism. Then, $\bigcap_{E' \in \mathcal{E}'} f_{\overrightarrow{\Omega'}}(E') \subseteq f_{\overrightarrow{\Omega'}}(\bigcap \mathcal{E}')$ for all $\mathcal{E}' \subseteq \mathcal{D}'$ with $(0 <)|\mathcal{E}'| < \kappa$.

[A.1] To be precise, we abuse the notation $f_{\overrightarrow{\Omega^*}}(\cdot)$ to denote the given property expressed by the operation f with respect to players' knowledge in $(\Omega^*, \mathcal{D}^*)$.

Proof of Lemma A.1. 1. Fix $\omega \in \Omega$ and $E' \in \mathcal{D}'$. Suppose that $\varphi(\omega) \in f_{\overrightarrow{\Omega'}}(E')$. Then, $\omega \in \varphi^{-1}(f_{\overrightarrow{\Omega'}}(E')) = f_{\overrightarrow{\Omega}}(\varphi^{-1}(E'))$. It follows from the assumption that $\omega \in g_{\overrightarrow{\Omega}}(\varphi^{-1}(E')) = \varphi^{-1}(g_{\overrightarrow{\Omega'}}(E'))$, i.e., $\varphi(\omega) \in g_{\overrightarrow{\Omega'}}(E')$.

2. (a) Fix $[e] \in \mathcal{D}^*$. If $\omega^* \in f_{\overrightarrow{\Omega^*}}([e])$ then there are a knowledge space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ with $D(\omega) = \omega^* \in f_{\overrightarrow{\Omega^*}}([e])$. Now, it follows from the previous argument that $\omega^* = D(\omega) \in g_{\overrightarrow{\Omega^*}}([e])$.

   (b) Fix $E' \in \mathcal{D}'$. If $\omega' \in f_{\overrightarrow{\Omega'}}(E')$ then there is $\omega \in \Omega$ with $\varphi(\omega) = \omega' \in f_{\overrightarrow{\Omega'}}(E')$. Now, it follows from the previous argument that $\omega' = \varphi(\omega) \in g_{\overrightarrow{\Omega'}}(E')$.

3. The proof is similar to that of (1). Fix $\omega \in \Omega$ and $E', F' \in \mathcal{D}'$ with $E' \subseteq F'$. Suppose that $\varphi(\omega) \in f_{\overrightarrow{\Omega'}}(E')$. Then, $\omega \in \varphi^{-1}(f_{\overrightarrow{\Omega'}}(E')) = f_{\overrightarrow{\Omega}}(\varphi^{-1}(E'))$. Since $\varphi^{-1}(E') \subseteq \varphi^{-1}(F')$, it follows from the assumption that $\omega \in f_{\overrightarrow{\Omega}}(\varphi^{-1}(F')) = \varphi^{-1}(f_{\overrightarrow{\Omega'}}(F'))$, i.e., $\varphi(\omega) \in f_{\overrightarrow{\Omega'}}(F')$.

4. The proof is similar to that of (2).

   (a) Fix $[e], [f] \in \mathcal{D}^*$ with $[e] \subseteq [f]$. If $\omega^* \in f_{\overrightarrow{\Omega^*}}([e])$ then there are a knowledge space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ with $D(\omega) = \omega^* \in f_{\overrightarrow{\Omega^*}}([e])$. Now, it follows from the previous argument that $\omega^* = D(\omega) \in f_{\overrightarrow{\Omega^*}}([f])$.

   (b) Fix $E', F' \in \mathcal{D}'$ with $E' \subseteq E'$. If $\omega' \in f_{\overrightarrow{\Omega'}}(E')$ then there is $\omega \in \Omega$ with $\varphi(\omega) = \omega' \in f_{\overrightarrow{\Omega'}}(E')$. Now, it follows from the previous argument that $\omega' = \varphi(\omega) \in f_{\overrightarrow{\Omega'}}(F')$.

5. The proof is similar to that of (1). Fix $\omega \in \Omega$ and $\mathcal{E}' \subseteq \mathcal{D}'$ with $|\mathcal{E}'| < \kappa$. Suppose that $\varphi(\omega) \in \bigcap_{E' \in \mathcal{E}'} f_{\overrightarrow{\Omega'}}(E')$. Then, $\omega \in \varphi^{-1}(f_{\overrightarrow{\Omega'}}(E')) = f_{\overrightarrow{\Omega}}(\varphi^{-1}(E'))$ for all $E' \in \mathcal{E}'$, i.e., $\omega \in \bigcap_{E' \in \mathcal{E}'} f_{\overrightarrow{\Omega}}(\varphi^{-1}(E'))$. Now, it follows from the assumption that $\omega \in f_{\overrightarrow{\Omega}}(\bigcap_{E' \in \mathcal{E}'} \varphi^{-1}(E')) = f_{\overrightarrow{\Omega}}(\varphi^{-1}(\bigcap \mathcal{E}')) = \varphi^{-1}(f_{\overrightarrow{\Omega'}}(\bigcap \mathcal{E}'))$, i.e., $\varphi(\omega) \in f_{\overrightarrow{\Omega'}}(\bigcap \mathcal{E}')$.

6. The proof is similar to that of (2).

   (a) Fix $\mathcal{E}^* \subseteq \mathcal{D}^*$ with $|\mathcal{E}^*| < \kappa$. If $\omega^* \in \bigcap_{[e] \in \mathcal{E}^*} f_{\overrightarrow{\Omega^*}}([e])$ then there are a knowledge space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ with $D(\omega) = \omega^* \in \bigcap_{[e] \in \mathcal{E}^*} f_{\overrightarrow{\Omega^*}}([e])$. Now, it follows from the previous argument that $\omega^* = D(\omega) \in f_{\overrightarrow{\Omega^*}}(\bigcap \mathcal{E}^*)$.

   (b) Suppose that φ is surjective. Fix $\mathcal{E}' \subseteq \mathcal{D}'$ with $|\mathcal{E}'| < \kappa$. If $\omega' \in \bigcap_{E' \in \mathcal{E}'} f_{\overrightarrow{\Omega'}}(E')$ then there is $\omega \in \Omega$ with $\varphi(\omega) = \omega' \in \bigcap_{E' \in \mathcal{E}'} f_{\overrightarrow{\Omega'}}(E')$. Now, it follows from the previous argument that $\omega' = \varphi(\omega) \in f_{\overrightarrow{\Omega'}}(\bigcap \mathcal{E}')$.

$\square$

We also establish the following lemma in order to assert the preservation of the Kripke property.

Lemma A.2 (Preservation of the Kripke Property)**.** *Let $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ be a knowledge morphism between knowledge spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$. For each $\omega \in \Omega$, we have $\varphi(b_{K_i}(\omega)) \subseteq b'_{K'_i}(\varphi(\omega))$, where recall that*

$$b_{K_i}(\omega) = \bigcap\{E \in \mathcal{D} \mid \omega \in K_i(E)\} \text{ and } b'_{K'_i}(\varphi(\omega)) = \bigcap\{E' \in \mathcal{D}' \mid \varphi(\omega) \in K'_i(E')\}.$$

Proof of Lemma A.2. Fix $\omega \in \Omega$. If $\varphi(\tilde{\omega}) \notin b'_{K'_i}(\varphi(\omega))$, then there is $E' \in \mathcal{D}'$ such that $\varphi(\omega) \in K'_i(E')$ and $\varphi(\tilde{\omega}) \notin K'_i(E')$. Since φ is a knowledge morphism, $\omega \in K_i(\varphi^{-1}(E'))$ and $\tilde{\omega} \notin K_i(\varphi^{-1}(E'))$. This implies that $\tilde{\omega} \notin b_{K_i}(\omega)$. This yields the desired expression.

In an alternative way, we can directly assert the desired expression as follows.

$$\begin{aligned} b'_{K'_i}(\varphi(\omega)) = \bigcap\{E' \in \mathcal{D}' \mid \omega \in K_i(\varphi^{-1}(E'))\} &\supseteq \bigcap_{E' \in \mathcal{D}' : \omega \in K_i(\varphi^{-1}(E'))} \varphi(\varphi^{-1}(E')) \\ &\supseteq \varphi(\bigcap_{E' \in \mathcal{D}' : \omega \in K_i(\varphi^{-1}(E'))} \varphi^{-1}(E')) \supseteq \varphi(b_{t_i}(\omega)). \end{aligned}$$

□

Now, we establish Lemma 3.

Proof of Lemma 3. Fix $i \in I$. We prove the statement in the following three steps. First, we show that the operator $K_i^* : \mathcal{D}^* \to \mathcal{D}^*$ is well defined, i.e., $K_i^*([e]) = K_i^*([f])$ for any $e, f \in \mathcal{D}$ with $[e] = [f]$.

Let $e, f \in \mathcal{D}$ be such that $[e] = [f]$. If $\omega^* \in K_i^*([e]) = [k_i(e)]$, then there are a knowledge space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $D(\omega) \in [k_i(e)]$, i.e., $(1, k_i(e)) \in D(\omega)$. Thus, we have $\omega \in [\![k_i(e)]\!]_{\overrightarrow{\Omega}} = K_i([\![e]\!]_{\overrightarrow{\Omega}}) = K_i(D^{-1}([e]))$. Since $[e] = [f]$, we obtain $\omega \in [\![k_i(f)]\!]_{\overrightarrow{\Omega}}$. That is, we have $\omega^* = D(\omega) \in [k_i(f)] = K_i^*([f])$. By changing the role of e and f, we conclude that $K_i^*([e]) = K_i^*([f])$.

Second, for any $[e] \in \mathcal{D}^*$, we have

$$K_i(D^{-1}([e])) = K_i([\![e]\!]_{\overrightarrow{\Omega}}) = [\![k_i(e)]\!]_{\overrightarrow{\Omega}} = D^{-1}([k_i(e)]) = D^{-1}(K_i^*([e])).$$

Third, we show that each K_i^* inherits all the properties imposed in the given category of knowledge spaces.

1. For No-Contradiction Axiom, apply Lemma A.1 (2a) by taking $(f_{\overrightarrow{\Omega}}, g_{\overrightarrow{\Omega}}) = (K_i(\emptyset), \emptyset)$.
2. For Consistency, apply Lemma A.1 (2a) by taking $(f_{\overrightarrow{\Omega}}, g_{\overrightarrow{\Omega}}) = (K_i(\cdot) \cap (\neg K_i)(\cdot), \emptyset)$.
3. For Truth Axiom, apply Lemma A.1 (2a) by taking $(f_{\overrightarrow{\Omega}}, g_{\overrightarrow{\Omega}}) = (K_i(\cdot), \mathrm{id}_{\mathcal{D}}(\cdot))$.
4. For Necessitation, apply Lemma A.1 (2a) by taking $(f_{\overrightarrow{\Omega}}, g_{\overrightarrow{\Omega}}) = (\Omega, K_i(\Omega))$.

5. For Positive Introspection, apply Lemma A.1 (2a) by taking $(f_{\overrightarrow{\Omega}}, g_{\overrightarrow{\Omega}}) = (K_i(\cdot), K_i K_i(\cdot))$.

6. For Negative Introspection, apply Lemma A.1 (2a) by taking $(f_{\overrightarrow{\Omega}}, g_{\overrightarrow{\Omega}}) = (K_i(\cdot), K_i(\neg K_i)(\cdot))$.

7. For Monotonicity, apply Lemma A.1 (4a) by taking $f_{\overrightarrow{\Omega}} = K_i(\cdot)$.

8. For Non-empty λ-Conjunction, apply Lemma A.1 (6a) by taking $f_{\overrightarrow{\Omega}}(\cdot) = K_i(\cdot)$.

9. Finally, consider the Kripke property. Note that, by Lemma A.2, we have $D(b_{K_i}(\omega)) \subseteq b^*_{K_i^*}(D(\omega))$ for each $\omega \in \Omega$. Suppose that $b^*_{K_i^*}(\omega^*) \subseteq [e]$. There are a knowledge space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $\omega^* = D(\omega)$. Then, we have

$$b_{K_i}(\omega) \subseteq D^{-1}(D(b_{K_i}(\omega))) \subseteq D^{-1}(b^*_{K_i^*}(\omega^*)) \subseteq D^{-1}([e]).$$

By the Kripke property of $\overrightarrow{\Omega}$, it follows that $\omega \in K_i(D^{-1}([e])) = D^{-1}(K_i^*([e]))$, i.e., $\omega^* = D(\omega) \in K_i^*([e])$.

$\square$

We provide the following two results. The first is a lemma regarding the preservation of properties of knowledge under surjective mappings.

Lemma A.3 (Surjection Preserves Properties of Knowledge). *Let $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$ be knowledge spaces which may reside in different classes of knowledge spaces. Let $\varphi : \Omega \to \Omega'$ be a surjective mapping such that $K_i(\varphi^{-1}(E')) = \varphi^{-1}(K_i'(E'))$ for all $E' \in \mathcal{D}'$. Then, the knowledge operator K_i' inherits all the properties of K_i.*

Proof of Lemma A.3. The proof is similar to the argument in the proof of Lemma 3 (i.e., applications of Lemmas A.1 and A.2), and hence omitted. $\square$

Second, we provide a sufficient condition for K_i^* to violate some property of knowledge if it is not assumed in the given category of knowledge spaces.

Proposition A.2 (When Does Candidate Universal Space Violate Properties of Knowledge?).

1. *Suppose that a property of knowledge is expressed by $f_{\overrightarrow{\Omega}}(\cdot) \subseteq g_{\overrightarrow{\Omega}}(\cdot)$ for each knowledge space $\overrightarrow{\Omega}$. Suppose that there is a knowledge space $\overrightarrow{\Omega}$ which violates this property with respect to some $e \in \mathcal{L}$. That is, $f_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) \not\subseteq g_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}})$. Then, we have $f_{\overrightarrow{\Omega^*}}([e]) \not\subseteq g_{\overrightarrow{\Omega^*}}([e])$.*

2. *Suppose that a property of knowledge is expressed by the monotonicity of $f_{\overrightarrow{\Omega}}(\cdot)$ for each knowledge space $\overrightarrow{\Omega}$. Suppose that there is a knowledge space $\overrightarrow{\Omega}$ which violates this property with respect to some expressions $e, f \in \mathcal{L}$ with $[\![e]\!]_{\overrightarrow{\Omega}} \subseteq [\![f]\!]_{\overrightarrow{\Omega}}$. That is, $f_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) \not\subseteq f_{\overrightarrow{\Omega}}([\![f]\!]_{\overrightarrow{\Omega}})$. Then, we have $f_{\overrightarrow{\Omega^*}}([e]) \not\subseteq f_{\overrightarrow{\Omega^*}}([f])$.*

3. *Suppose that a property of knowledge is expressed by the non-empty λ-conjunction of $f_{\overrightarrow{\Omega}}(\cdot)$ for each knowledge space $\overrightarrow{\Omega}$. Suppose that there is a knowledge space $\overrightarrow{\Omega}$ which violates this property with respect to some set of expressions $\mathcal{E}(\subseteq \mathcal{L})$ with $0 < |\mathcal{E}| < \lambda(\leq \kappa)$. That is, $\bigcap_{e\in\mathcal{E}} f_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) \not\subseteq f_{\overrightarrow{\Omega}}(\bigcap_{e\in\mathcal{E}}[\![e]\!]_{\overrightarrow{\Omega}})$. Then, we have $\bigcap_{e\in\mathcal{E}} f_{\overrightarrow{\Omega^*}}([e]) \not\subseteq f_{\overrightarrow{\Omega^*}}(\bigcap_{e\in\mathcal{E}}[e])$.*

Proof of Proposition A.2. 1. Suppose that there are a knowledge space $\overrightarrow{\Omega}$ and an expression $e \in \mathcal{L}$ such that $f_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) \not\subseteq g_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}})$. Then, there is $\omega \in \Omega$ such that $\omega \in f_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}})$ and $\omega \notin g_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}})$. Since $f_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) = D^{-1}(f_{\overrightarrow{\Omega^*}}([e]))$ and $g_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) = D^{-1}(g_{\overrightarrow{\Omega^*}}([e]))$, we obtain $D(\omega) \in f_{\overrightarrow{\Omega^*}}([e])$ and $D(\omega) \notin g_{\overrightarrow{\Omega^*}}([e])$.

2. Suppose that there are a knowledge space $\overrightarrow{\Omega}$ and expressions $e, f \in \mathcal{L}$ such that $f_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) \not\subseteq f_{\overrightarrow{\Omega}}([\![f]\!]_{\overrightarrow{\Omega}})$. Then, there is $\omega \in \Omega$ such that $\omega \in f_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}})$ and $\omega \notin f_{\overrightarrow{\Omega}}([\![f]\!]_{\overrightarrow{\Omega}})$. Now, we obtain $D(\omega) \in f_{\overrightarrow{\Omega^*}}([e])$ and $D(\omega) \notin f_{\overrightarrow{\Omega^*}}([f])$.

3. Suppose that there are a knowledge space $\overrightarrow{\Omega}$ and a set of expressions $\mathcal{E}(\subseteq \mathcal{L})$ such that $\bigcap_{e\in\mathcal{E}} f_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) \not\subseteq f_{\overrightarrow{\Omega}}(\bigcap_{e\in\mathcal{E}}[\![e]\!]_{\overrightarrow{\Omega}})$ and $0 < |\mathcal{E}| < \lambda(\leq \kappa)$. Then, there is $\omega \in \Omega$ such that $\omega \in \bigcap_{e\in\mathcal{E}} f_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) = D^{-1}(\bigcap_{e\in\mathcal{E}} f_{\overrightarrow{\Omega^*}}([e]))$ and

$$\omega \notin f_{\overrightarrow{\Omega}}(\bigcap_{e\in\mathcal{E}}[\![e]\!]_{\overrightarrow{\Omega}}) = f_{\overrightarrow{\Omega}}([\![\bigwedge \mathcal{E}]\!]_{\overrightarrow{\Omega}}) = D^{-1}(f_{\overrightarrow{\Omega^*}}([\bigwedge \mathcal{E}])) = D^{-1}(f_{\overrightarrow{\Omega^*}}(\bigcap_{e\in\mathcal{E}}[e])).$$

Now, we obtain $D(\omega) \in \bigcap_{e\in\mathcal{E}} f_{\overrightarrow{\Omega^*}}([e])$ and $D(\omega) \notin f_{\overrightarrow{\Omega^*}}(\bigcap_{e\in\mathcal{E}}[e])$.

□

Proof of Lemma 4. For any $\omega^* \in \Omega^*$, there are a knowledge space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $\omega^* = D(\omega)$. Choose such $\overrightarrow{\Omega}$ and ω and define $\Theta^*(\omega^*) := \Theta(\omega)$, where $(0, \Theta(\omega)) \in D(\omega)$.

It can be easily seen that $\Theta^* : \Omega^* \to S$ is well defined (i.e., Θ^* does not depend on a particular choice of $\overrightarrow{\Omega}$ and ω with $\omega^* = D(\omega)$). Indeed, suppose that $\omega^* = D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega'}}(\omega')$ for some $\omega \in \Omega$ and $\omega' \in \Omega'$, where $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$ are knowledge spaces. Then, $(0, \Theta(\omega)) = (0, \Theta'(\omega'))$, i.e., $\Theta(\omega) = \Theta'(\omega')$.

Finally, take $E \in \mathcal{A}_S$. We show $(\Theta^*)^{-1}(E) = [E] \in \mathcal{D}^*$ as follows:

$$\begin{aligned}\omega^* \in (\Theta^*)^{-1}(E) \text{ iff } \Theta^*(\omega^*) = \Theta^*(D(\omega)) \in E \text{ iff } \Theta(\omega) \in E \\ \text{iff } \omega \in \Theta^{-1}(E) = [\![E]\!]_{\overrightarrow{\Omega}} \text{ iff } E \in D(\omega) = \omega^* \text{ iff } \omega^* \in [E],\end{aligned}$$

where $\overrightarrow{\Omega}$ and $\omega \in \Omega$ satisfy $\omega^* = D(\omega)$. □

In order to prove Proposition 1, consider the following lemma.

Lemma A.4 (Logical Properties of Each State)**.** *Fix $\omega^* \in \Omega^*$.*

1. *For each $e \in \mathcal{L}$, $(1, e) \notin \omega^*$ iff $(1, (\neg e)) \in \omega^*$.*

2. *For any $\mathcal{E}$ with $\mathcal{E} \subseteq \mathcal{L}$ and $|\mathcal{E}| < \kappa$, we have $(1, \bigwedge \mathcal{E}) \in \omega^*$ iff $(1, e) \in \omega^*$ for all $e \in \mathcal{E}$.*

Proof of Lemma A.4. Fix $\omega^* \in \Omega^*$.

1. Suppose that $(1, e) \in \omega^*$, i.e., $\omega^* \in [e]$. Now, if it were the case that $(1, (\neg e)) \in \omega^*$, then we also have $\omega^* \in [\neg e] = \neg[e]$, a contradiction. Conversely, suppose that $(1, e) \notin \omega^*$. Then, $\omega^* \notin [e]$, i.e., $\omega^* \in \neg[e] = [\neg e]$. Thus, we have $(1, (\neg e)) \in \omega^*$.

2. Let $\mathcal{E}$ be a subset of $\mathcal{L}$ with $|\mathcal{E}| < \kappa$. If $(1, e) \in \omega^*$ (i.e., $\omega^* \in [e]$) for all $e \in \mathcal{E}$, then we obtain $\omega^* \in \bigcap_{e \in \mathcal{E}}[e] = [\bigwedge \mathcal{E}]$. Thus, we have $(1, \bigwedge \mathcal{E}) \in \omega^*$. Conversely, suppose that $(1, \bigwedge \mathcal{E}) \in \omega^*$, i.e., $\omega^* \in [\bigwedge \mathcal{E}] = \bigcap_{e \in \mathcal{E}}[e]$. It is clear that $\omega^* \in [e]$, i.e., $(1, e) \in \omega^*$, for all $e \in \mathcal{E}$.

$\square$

Proof of Proposition 1. Fix $i \in I$.

1. The assertion clearly follows from Lemma A.4.

2. Suppose that $(1, k_i(e)) \notin \omega^*$ and $(1, k_i(\neg e)) \notin \omega^*$. Then, we have $\omega^* \in [(\neg k_i)(e)] \cap [(\neg k_i)(\neg e)] = [(\neg k_i)(e) \wedge (\neg k_i)(\neg e)]$. This implies $(1, (\neg k_i)(e) \wedge (\neg k_i)(\neg e)) \in \omega^*$.

   Next, it is clear that if $(1, k_i(e)) \in \omega^*$ then $(1, (\neg k_i)(e) \wedge (\neg k_i)(\neg e)) \notin \omega^*$. Likewise, $(1, k_i(\neg e)) \in \omega^*$ then $(1, (\neg k_i)(e) \wedge (\neg k_i)(\neg e)) \notin \omega^*$. Also, if $(1, (\neg k_i)(e) \wedge (\neg k_i)(\neg e)) \in \omega^*$ then $(1, k_i(e)) \notin \omega^*$ and $(1, k_i(\neg e)) \notin \omega^*$. Without Consistency, however, it might be possible that $(1, k_i(e)) \in \omega^*$ and $(1, k_i(\neg e)) \in \omega^*$.

   Next, assume that i's knowledge satisfies Consistency. Suppose to the contrary that $(1, k_i(e)) \in \omega^*$ and $(1, k_i(\neg e)) \in \omega^*$. Then, we get $\omega \in K_i^*([e]) \cap K_i^*(\neg[e]) = \emptyset$, a contradiction.

   Conversely, assume that exactly one of the three conditions holds. If $\omega^* \in K_i^*([e])$ then $(1, k_i(e)) \in \omega^*$. Then, we get $(1, k_i(\neg e)) \notin \omega^*$, i.e., $\omega^* \in (\neg K_i^*)([\neg e]) = (\neg K_i^*)([e]^c)$, establishing Consistency.

3. The first assertion clearly follows from Lemma A.4. The second condition follows because we have

$$(1, (\neg k_i)(e) \wedge (\neg k_i)(\neg k_i)(e)) \in \omega^* \text{ iff } \omega^* \in (\neg K_i^*)([e]) \cap (\neg K_i^*)(\neg K_i^*)([e]).$$

$\square$

Proof of Lemma 5. First, we show that there is a unique $s \in S$ such that $\omega^* \cap \{0\} \times S = D(\omega^*) \cap \{0\} \times S = \{(0, s)\}$. Second, in a similar way to HS [39, Lemma 4.6], Meier [55, Lemma 6], and Meier [56, Lemma 4], we show that $[\![e]\!]_{\overrightarrow{\Omega^*}} = [e]$ for every $e \in \mathcal{L}$ by induction on the formation of expressions. Once these assertions are established, we have $\omega^* = \{s\} \sqcup \{e \in \mathcal{L} \mid (1, e) \in \omega^*\} = \{s\} \sqcup \{e \in \mathcal{L} \mid \omega^* \in [e]\} = \{s\} \sqcup \{e \in \mathcal{L} \mid \omega^* \in [\![e]\!]_{\overrightarrow{\Omega^*}}\} = D(\omega^*)$ for any $\omega^* \in \Omega^*$, where s is a unique element associated with ω^* and $D(\omega^*)$.

In the first step, assume that $(0, s) \in \omega^*$ and $(0, s') \in D(\omega^*)$. There are a knowledge space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $s = \Theta(\omega)$ and $s' = \Theta^*(\omega^*) = \Theta^*(D(\omega))$. Now, since $\Theta(\omega) = \Theta^*(D(\omega))$, we obtain $s = s'$. Note that the argument does not depend on a particular choice of knowledge spaces.

In the second step, we show by induction that $[\![e]\!]_{\overrightarrow{\Omega^*}} = [e]$ for every $e \in \mathcal{L}$. For any $E \in \mathcal{A}_S$, we have the following:

$$\begin{aligned} &\omega^* \in [\![E]\!]_{\overrightarrow{\Omega^*}} (:= (\Theta^*)^{-1}(E)) \text{ iff } \Theta^*(\omega^*) \in E \text{ iff } \Theta^*(D(\omega)) = \Theta(\omega) \in E \\ &\text{iff } \omega \in \Theta^{-1}(E) = [\![E]\!]_{\overrightarrow{\Omega}} \text{ iff } (1, E) \in D(\omega) \text{ iff } \omega^* = D(\omega) \in [E], \end{aligned}$$

where $\overrightarrow{\Omega}$ and $\omega \in \Omega$ satisfy $\omega^* = D(\omega)$.

Next, let $\mathcal{E}$ be a non-empty set of expressions with $|\mathcal{E}| < \kappa$. Assume the induction hypothesis that $[\![e]\!]_{\overrightarrow{\Omega^*}} = [e]$ for all $e \in \mathcal{E}$. Then, we have $[\![\bigwedge \mathcal{E}]\!]_{\overrightarrow{\Omega^*}} = \bigcap_{e \in \mathcal{E}} [\![e]\!]_{\overrightarrow{\Omega^*}} = \bigcap_{e \in \mathcal{E}} [e] = [\bigwedge \mathcal{E}]$, where the last equality follows from the proof of Lemma 2.

Next, assume the induction hypothesis that $[\![e]\!]_{\overrightarrow{\Omega^*}} = [e]$. Then, by definition, we have $[k_i(e)] = K_i^*([e]) = K_i^*([\![e]\!]_{\overrightarrow{\Omega^*}}) = [\![k_i(e)]\!]_{\overrightarrow{\Omega^*}}$. It can also be seen that $[(\neg e)] = \neg[e] = \neg[\![e]\!]_{\overrightarrow{\Omega^*}} = [\![\neg e]\!]_{\overrightarrow{\Omega^*}}$. The proof is complete. $\square$

Proof of Theorem 1. Recall that we have shown that $\overrightarrow{\Omega^*} = \langle (\Omega^*, \mathcal{D}^*), (K_i^*)_{i \in I}, \Theta^* \rangle$ is a knowledge space of I on $(S, \mathcal{A}_S)$ of the given category (Lemmas 2, 3, and 4) and that for any knowledge space $\overrightarrow{\Omega} = \langle (\Omega, \mathcal{D}), (K_i)_{i \in I}, \Theta \rangle$ of I on $(S, \mathcal{A}_S)$, the description map $D_{\overrightarrow{\Omega}} : \Omega \to \Omega^*$ is a knowledge morphism (Lemmas 2, 3, and 4).

Thus, we need only to show that $D_{\overrightarrow{\Omega}}$ is a unique knowledge morphism from $\overrightarrow{\Omega}$ into $\overrightarrow{\Omega^*}$. Suppose that $\varphi : \Omega \to \Omega^*$ is a knowledge morphism. Then, it follows from Corollary 1 that $D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega^*}}(\varphi(\omega))$ for any $\omega \in \Omega$. On the other hand, it follows from Lemma 5 that the description map $D_{\overrightarrow{\Omega^*}} : \Omega^* \to \Omega^*$ is the identity on Ω^*. Thus, we get $D_{\overrightarrow{\Omega}} = \varphi$. The proof is complete. $\square$

Proof of Remark 2. Here, we only prove the second statement as the first one is clear. Fix a knowledge space $\overrightarrow{\Omega}$. We let $\Omega' := D(\Omega)$. We denote by $i_{\Omega'}$ the inclusion map from Ω' into Ω^*. We define the knowledge space $\overrightarrow{\Omega'}$ as follows. First, we let $\mathcal{D}' := \{(i_{\Omega'})^{-1}([e]) \mid [e] \in \mathcal{D}^*\}$. By construction, $(\Omega', \mathcal{D}')$ is a κ-complete algebra. Second, we let $\Theta' = \Theta^*|_{\Omega'} (= \Theta^* \circ i_{\Omega'})$. By construction, we have $(\Theta')^{-1}(E) = (i_{\Omega'})^{-1}((\Theta^*)^{-1}(E)) \in \mathcal{D}'$ for all $E \in \mathcal{A}_S$. Third, we define $K_i'((i_{\Omega'})^{-1}([e])) := (i_{\Omega'})^{-1}(K_i^*([e]))$ for all $[e] \in \mathcal{D}^*$. We show that this is well defined. Suppose that

$(i_{\Omega'})^{-1}([e]) = (i_{\Omega'})^{-1}([f])$ for some $e, f \in \mathcal{L}$. If $\omega^* \in (i_{\Omega'})^{-1}(K_i^*([e])) = K_i^*([e]) \cap D(\Omega)$, then there is $\omega \in \Omega$ such that $\omega \in D^{-1}(K_i^*([e])) = K_i(D^{-1}([e])) = K_i(D^{-1}(i_{\Omega'})^{-1}([e])) = K_i(D^{-1}(i_{\Omega'})^{-1}([f]))$. Then, it can be easily seen that $\omega^* \in (i_{\Omega'})^{-1}(K_i^*([f]))$. By changing the role of e and f, we obtain $(i_{\Omega'})^{-1}(K_i^*([e])) = (i_{\Omega'})^{-1}(K_i^*([f]))$. Thus, K_i' is well defined.

Now, we consider $D' : \Omega \to \Omega'$. By construction, it is surjective. Also, we obtain

$$\begin{aligned}(D')^{-1}K_i'((i_{\Omega'})^{-1}([e])) &= (D')^{-1}(i_{\Omega'})^{-1}K_i^*([e]) = D^{-1}(K_i^*([e])) \\ &= K_i(D^{-1}([e])) = K_i((D')^{-1}((i_{\Omega'})^{-1}([e]))).\end{aligned}$$

It follows (i) that $\overrightarrow{\Omega'}$ inherits the properties of knowledge imposed in $\overrightarrow{\Omega}$ (recall Lemma A.3) and (ii) that $D' : \Omega \to \Omega'$ is a knowledge morphism.

By construction, $\overrightarrow{\Omega'}$ is a knowledge subspace of $\overrightarrow{\Omega^*}$, because the inclusion map $i_{\Omega'}$ is a knowledge morphism. □

Proof of Proposition 2.

1. It is sufficient to show that if Φ is satisfiable then it is satisfiable in $\overrightarrow{\Omega^*}$. Suppose that Φ is satisfiable, i.e., there are a knowledge space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $[\![f]\!]_{\overrightarrow{\Omega}} = D^{-1}([f])$ for all $f \in \Phi$. Then, we have $D(\omega) \in [f] = [\![f]\!]_{\overrightarrow{\Omega^*}}$ for all $f \in \Phi$.
2. It is enough to show that $\Phi \models_{\overrightarrow{\Omega^*}} e$ implies $\Phi \models e$. Let $\overrightarrow{\Omega}$ be any knowledge space. If $\omega \in [\![f]\!]_{\overrightarrow{\Omega}} = D^{-1}([f])$ for all $f \in \Phi$, then again $D(\omega) \in [f] = [\![f]\!]_{\overrightarrow{\Omega^*}}$ for all $f \in \Phi$. By assumption, we have $D(\omega) \in [\![e]\!]_{\overrightarrow{\Omega^*}} = [e]$, i.e., $\omega \in D^{-1}([e]) = [\![e]\!]_{\overrightarrow{\Omega}}$. Thus, $\Phi \models e$.
3. This assertion can be seen as a special case of the last assertion. Indeed, suppose that $\Omega^* = [\![e]\!]_{\overrightarrow{\Omega^*}} = [e]$. For any $\overrightarrow{\Omega}$ and $\omega \in \Omega$, we have $D(\omega) \in [e]$ and thus $\omega \in D^{-1}([e]) = [\![e]\!]_{\overrightarrow{\Omega}}$. Hence, $\Omega = [\![e]\!]_{\overrightarrow{\Omega}}$.

□

Proof of Proposition 3. We divide the proof into the two steps.

Step 1. We show that the mapping defined in the statement of the proposition is injective. Suppose that $\Theta^*(\omega^*) = \Theta^*(\tilde{\omega}^*)$ and

$$\{e \in \mathcal{L} \mid \omega^* \in K_i^*([e])\} = \{e \in \mathcal{L} \mid \tilde{\omega}^* \in K_i^*([e])\} \text{ for each } i \in I,$$

where note that $[\cdot] = [\![\cdot]\!]_{\overrightarrow{\Omega^*}}$.

First, we have $(0, \Theta^*(\omega^*)) \in \omega^*$ and $(0, \Theta^*(\tilde{\omega}^*)) \in \tilde{\omega}^*$. Thus, ω^* and $\tilde{\omega}^*$ contain the same unique element of S.

Second, we show by induction that ω^* and $\tilde{\omega}^*$ contain the same set of expressions. For any $E \in \mathcal{A}_S$ with $(1, E) \in \omega^*$ (i.e., $\omega^* \in [E] = (\Theta^*)^{-1}(E)$), we have $\Theta^*(\tilde{\omega}^*) = \Theta^*(\omega^*) \in E$ and thus $(1, E) \in \tilde{\omega}^*$. The converse is also true.

Next, we have $(1, (\neg e)) \in \omega^*$ iff $(1, e) \notin \omega^*$ iff $(1, e) \notin \tilde{\omega}^*$ iff $(1, (\neg e)) \in \tilde{\omega}^*$.

Let $\mathcal{E}$ be a subset of $\mathcal{L}$ with $(0 <)|\mathcal{E}| < \kappa$. Then, $(1, \bigwedge \mathcal{E}) \in \omega^*$ iff $(1, e) \in \omega^*$ for all $e \in \mathcal{E}$ iff $(1, e) \in \tilde{\omega}^*$ for all $e \in \mathcal{E}$ iff $(1, \bigwedge \mathcal{E}) \in \tilde{\omega}^*$.

Fix $i \in I$. We have $(1, k_i(e)) \in \omega^*$ iff $\omega^* \in K_i^*([e])$ iff $\tilde{\omega}^* \in K_i^*([e])$ iff $(1, k_i(e)) \in \tilde{\omega}^*$. Thus, the induction is complete. We have $\omega^* = \tilde{\omega}^*$.

Step 2. Next, we show that the mapping defined in the statement of the proposition is surjective. Let $\overrightarrow{\Omega}$ be a knowledge space, and let $\omega \in \Omega$ be a state. Take any $(\Theta(\omega), (\{e \in \mathcal{L} \mid \omega \in K_i([\![e]\!]_{\overrightarrow{\Omega}})\})_{i \in I})$. Then, it follows from follows from Lemmas 3 and 5 that

$$\begin{aligned}(\Theta(\omega), (\{e \in \mathcal{L} \mid \omega \in K_i([\![e]\!]_{\overrightarrow{\Omega}})\})_{i \in I}) &= (\Theta^*(D(\omega)), (\{e \in \mathcal{L} \mid D(\omega) \in K_i^*([\![e]\!]_{\overrightarrow{\Omega^*}})\})_{i \in I}) \\ &= (\Theta^*(D(\omega)), (\{e \in \mathcal{L} \mid D(\omega) \in K_i^*([e])\})_{i \in I}).\end{aligned}$$

The proof is complete. □

A.2.2 Section 3.2 (Proof of Theorem 2)

Step 1. The proof consists of the following two steps. In the first step, we show that Ω^* satisfies all the properties mentioned in Theorem 2.

1. Fix $\omega^* \in \Omega^*$. Since Conditions (d) to (f) follow from Lemma A.4, we establish (a)-(c).
   (a) There is a unique $s = \Theta^*(\omega^*) \in S$ such that $(0, s) \in \omega^*$. For any $E \in \mathcal{A}_S$ with $s \in E$, we have $\omega^* \in (\Theta^*)^{-1}(E) = [E]$, i.e., $(1, E) \in \omega^*$.
   (b) It is enough to show that each of the following expressions is valid in $\overrightarrow{\Omega^*}$.
      i. Suppose that i's knowledge satisfies No-Contradiction Axiom. Then, it can be easily seen that $[\![\emptyset \leftrightarrow k_i(\emptyset)]\!]_{\overrightarrow{\Omega^*}} = (\neg K_i^*)(\emptyset) = \Omega^*$, and thus $(\emptyset \leftrightarrow k_i(\emptyset))$ is valid in $\overrightarrow{\Omega^*}$.
      ii. If i's knowledge satisfies Consistency, then we have
      $$[\![k_i(e) \to (\neg k_i)(\neg e)]\!]_{\overrightarrow{\Omega^*}} = (\neg K_i^*([\![e]\!]_{\overrightarrow{\Omega^*}})) \cup (\neg K_i^*)(\neg [\![e]\!]_{\overrightarrow{\Omega^*}}) = \Omega^*.$$
      iii. Suppose that i's knowledge satisfies Non-empty λ-Conjunction. Take any $\mathcal{E}$ such that $\mathcal{E} \subseteq \mathcal{L}$ and $0 < |\mathcal{E}| < \kappa$. Then, we have
      $$\left[\!\!\left[(\bigwedge_{e \in \mathcal{E}} k_i(e)) \to k_i(\bigwedge \mathcal{E})\right]\!\!\right]_{\overrightarrow{\Omega^*}} = \left(\neg \bigcap_{e \in \mathcal{E}} K_i^*([\![e]\!]_{\overrightarrow{\Omega^*}})\right) \cup K_i^*(\bigcap_{e \in \mathcal{E}} [\![e]\!]_{\overrightarrow{\Omega^*}}) = \Omega^*.$$
      iv. If Necessitation is imposed on player i, then $k_i(S)$ is valid because $[\![k_i(S)]\!]_{\overrightarrow{\Omega^*}} = K_i^*([\![S]\!]_{\overrightarrow{\Omega^*}}) = K_i^*(\Omega^*) = \Omega^*$.

v. If i's knowledge satisfies Truth Axiom, then we have

$$[\![k_i(e) \to e]\!]_{\overrightarrow{\Omega^*}} = (\neg K_i^*)([\![e]\!]_{\overrightarrow{\Omega^*}}) \cup [\![e]\!]_{\overrightarrow{\Omega^*}} = \Omega^*.$$

vi. If i's knowledge satisfies Positive Introspection, then we have

$$[\![k_i(e) \to k_i k_i(e)]\!]_{\overrightarrow{\Omega^*}} = (\neg K_i^*)([\![e]\!]_{\overrightarrow{\Omega^*}}) \cup K_i^* K_i^*([\![e]\!]_{\overrightarrow{\Omega^*}}) = \Omega.$$

vii. If i's knowledge satisfies Negative Introspection, then we obtain

$$[\![(\neg k_i)(e) \to k_i(\neg k_i)(e)]\!]_{\overrightarrow{\Omega^*}} = K_i^*([\![e]\!]_{\overrightarrow{\Omega^*}}) \cup K_i^*(\neg K_i^*)([\![e]\!]_{\overrightarrow{\Omega^*}}) = \Omega^*.$$

(c) Suppose that $(1, e) \in \omega^*$ and $(1, (e \to f)) \in \omega^*$. Then, we get $\omega^* \in [e]$ and $\omega^* \in [e \to f] = [(\neg e) \vee f] = \neg[e] \cup [f]$. Thus, we must have $\omega^* \in [f]$, i.e., $(1, f) \in \omega^*$.

2. We show that Ω^* satisfies the following three conditions.

(a) If $(e \leftrightarrow f)$ is valid, then we have $[\![e]\!]_{\overrightarrow{\Omega}} = [\![f]\!]_{\overrightarrow{\Omega}}$ for any $\overrightarrow{\Omega}$. Then, we have $K_i([\![e]\!]_{\overrightarrow{\Omega}}) = K_i([\![f]\!]_{\overrightarrow{\Omega}})$, i.e., $[\![k_i(e)]\!]_{\overrightarrow{\Omega}} = [\![k_i(f)]\!]_{\overrightarrow{\Omega}}$. Now, it is easy to see that $[\![k_i(e) \leftrightarrow k_i(f)]\!]_{\overrightarrow{\Omega}} = \Omega$. Since $\overrightarrow{\Omega}$ is arbitrary, $(k_i(e) \leftrightarrow k_i(f))$ is valid.

(b) Suppose that i's knowledge satisfies Monotonicity. The proof is similar to the above one. If $(e \to f)$ is valid, then we have $[\![e]\!]_{\overrightarrow{\Omega}} \subseteq [\![f]\!]_{\overrightarrow{\Omega}}$ for any $\overrightarrow{\Omega}$. Then, we have $K_i([\![e]\!]_{\overrightarrow{\Omega}}) \subseteq K_i([\![f]\!]_{\overrightarrow{\Omega}})$, i.e., $[\![k_i(e)]\!]_{\overrightarrow{\Omega}} \subseteq [\![k_i(f)]\!]_{\overrightarrow{\Omega}}$. Now, it is easy to see that $[\![k_i(e) \to k_i(f)]\!]_{\overrightarrow{\Omega}} = \Omega$. Since $\overrightarrow{\Omega}$ is arbitrary, $(k_i(e) \to k_i(f))$ is valid.

(c) Suppose that $\{f \in \mathcal{L} \mid (1, k_i(f)) \in \omega^*\} \models_{\overrightarrow{\Omega^*}} e$. Since $\omega^* \in K_i^*([\![f]\!]_{\overrightarrow{\Omega^*}})$ iff $(1, k_i(f)) \in \omega^*$, we obtain $\bigcap\{[\![f]\!] \in \mathcal{D}^* \mid \omega^* \in K_i^*([\![f]\!]_{\overrightarrow{\Omega^*}})\} \subseteq [\![e]\!]_{\overrightarrow{\Omega^*}}$. By the Kripke property of $\overrightarrow{\Omega^*}$, we obtain $\omega^* \in K_i^*([\![e]\!]_{\overrightarrow{\Omega^*}}) = [k_i(e)]$, i.e., $(1, k_i(e)) \in \omega^*$.

In sum, Ω^* satisfies all the properties mentioned in Theorem 2.

Step 2. In the second step, we show that $\Omega \subseteq \Omega^*$ for any set Ω satisfying conditions specified in Theorem 2. To that end, we introduce a knowledge structure on Ω and show that the description map $D : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ is an inclusion map.

Step 2.1. We start with defining a κ-complete algebra on Ω. We let $\mathcal{D} := \{[e]_{\overrightarrow{\Omega}} \in \mathcal{P}(\Omega) \mid e \in \mathcal{L}\}$, where $[e]_{\overrightarrow{\Omega}} := \{\omega \in \Omega \mid (1, e) \in \omega\}$. Note the following. First, $[\cdot]_{\overrightarrow{\Omega}} \in \mathcal{D}$ is different from $[\cdot] \in \mathcal{D}^*$. Second, we use the notation $[\cdot]_{\overrightarrow{\Omega}}$ rather than $[\cdot]_\Omega$ although we have not introduced a knowledge structure to Ω.

We show that $(\Omega, \mathcal{D})$ is a κ-complete algebra. First, we show that $[\emptyset]_{\overrightarrow{\Omega}} = \emptyset$. Suppose to the contrary that $\omega \in [\emptyset]_{\overrightarrow{\Omega}}$. Then, by definition, $(1, \emptyset) \in \omega$, which is impossible. Hence, $\emptyset = [\emptyset]_{\overrightarrow{\Omega}} \in \mathcal{D}$. Second, it is immediate that $[S]_{\overrightarrow{\Omega}} = \Omega$ and thus $\Omega = [S]_{\overrightarrow{\Omega}} \in \mathcal{D}$.

Third, we show that $[\neg e]_{\overrightarrow{\Omega}} = \neg[e]_{\overrightarrow{\Omega}}$. Indeed, if $\omega \in [\neg e]_{\overrightarrow{\Omega}}$ then $(1, (\neg e)) \in \omega$ and thus $(1, e) \notin \omega$. Hence, $\omega \in \neg[e]_{\overrightarrow{\Omega}}$. The converse is similarly established. Hence, $\mathcal{D}$ is closed under complementation.

Fourth, we show that $[\bigwedge \mathcal{E}]_{\overrightarrow{\Omega}} = \bigcap_{e \in \mathcal{E}} [e]_{\overrightarrow{\Omega}}$ for any $\mathcal{E}$ with $\mathcal{E} \subseteq \mathcal{L}$ and $(0 <)|\mathcal{E}| < \kappa$. If $\omega \in \bigcap_{e \in \mathcal{E}} [e]_{\overrightarrow{\Omega}}$, then $(1, e) \in \omega$ for all $e \in \mathcal{E}$. Hence, we obtain $(1, \bigwedge \mathcal{E}) \in \omega$, i.e., $\omega \in [\bigwedge \mathcal{E}]_{\overrightarrow{\Omega}}$. Conversely, suppose that $\omega \in [\bigwedge \mathcal{E}]_{\overrightarrow{\Omega}}$, i.e., $(1, \bigwedge \mathcal{E}) \in \omega$. Then, we have $(1, e) \in \omega$ (i.e., $\omega \in [e]_{\overrightarrow{\Omega}}$) for all $e \in \mathcal{E}$, that is, $\omega \in \bigcap_{e \in \mathcal{E}} [e]_{\overrightarrow{\Omega}}$. This completes the proof of the fact that $(\Omega, \mathcal{D})$ is a κ-complete algebra.

Step 2.2. Next, we define the mapping $\Theta : \Omega \to S$ which associates, with each state of the world ω, the unique state of nature $s \in S$ with $(0, s) \in \omega$. Then, the mapping $\Theta : \Omega \to S$ is a well-defined κ-measurable map such that $(\Theta)^{-1}(E) = [E]_{\overrightarrow{\Omega}}$ for each $E \in \mathcal{A}_S$.

Indeed, take $E \in \mathcal{A}_S$. If $\omega \in [E]_{\overrightarrow{\Omega}}$, then $(1, E) \in \omega$. Hence, $\Theta(\omega) \in E$, i.e., $\omega \in \Theta^{-1}(E)$. Conversely, if $\omega \in \Theta^{-1}(E)$ then $\Theta(\omega) \in E$, and thus $(1, E) \in \omega$. Hence, $\omega \in [E]_{\overrightarrow{\Omega}}$.

Step 2.3. Next, we define players' knowledge operators on $(\Omega, \mathcal{D})$. Fix $i \in I$. We let $K_i : \mathcal{D} \to \mathcal{D}$ be such that $K_i([e]_{\overrightarrow{\Omega}}) := [k_i(e)]_{\overrightarrow{\Omega}}$ for each $[e] \in \mathcal{D}$.

We first show that each K_i is well defined. Suppose that $[e]_{\overrightarrow{\Omega}} = [f]_{\overrightarrow{\Omega}}$. Then, it is clear that $[(e \leftrightarrow f)]_{\overrightarrow{\Omega}} = \Omega$. This implies that $[(k_i(e) \leftrightarrow k_i(f))]_{\overrightarrow{\Omega}} = \Omega$. Thus, we obtain $[k_i(e)] = [k_i(f)]$.

Second, we show below that K_i inherits the properties of knowledge imposed in the given category.

1. Suppose that i's knowledge satisfies No-Contradiction Axiom in the given category. If $\omega \in K_i([\emptyset]_{\overrightarrow{\Omega}}) = [k_i(\emptyset)]_{\overrightarrow{\Omega}}$, then $(1, k_i(\emptyset)) \in \omega$. Since $(\emptyset \leftrightarrow k_i(\emptyset)) \in \omega$, it follows that $(1, \emptyset) \in \omega$, which is impossible.

2. Consider Consistency of i's knowledge. If $\omega \in K_i([e]_{\overrightarrow{\Omega}}) = [k_i(e)]_{\overrightarrow{\Omega}}$ then $(1, k_i(e)) \in \omega$. Now, since $(1, (k_i(e) \to (\neg k_i)(\neg e))) \in \omega$, it follows that $(1, (\neg k_i)(\neg e))) \in \omega$, i.e., $\omega \in [(\neg k_i)(\neg e)]_{\overrightarrow{\Omega}} = (\neg K_i)(\neg [e]_{\overrightarrow{\Omega}})$.

3. Consider Necessitation of i's knowledge. We have $\Omega = [k_i(S)]_{\overrightarrow{\Omega}} = K_i([S]_{\overrightarrow{\Omega}}) = K_i(\Omega)$.

4. Consider Monotonicity of i's knowledge. Take $[e]_{\overrightarrow{\Omega}}, [f]_{\overrightarrow{\Omega}} \in \mathcal{D}$ with $[e]_{\overrightarrow{\Omega}} \subseteq [f]_{\overrightarrow{\Omega}}$. Then, observe that $[e \to f]_{\overrightarrow{\Omega}} = \Omega$. Now, we obtain $[k_i(e) \to k_i(f)]_{\overrightarrow{\Omega}} = \Omega$, i.e., $[k_i(e)]_{\overrightarrow{\Omega}} \subseteq [k_i(f)]_{\overrightarrow{\Omega}}$. Thus, we have $K_i([e]_{\overrightarrow{\Omega}}) \subseteq K_i([f]_{\overrightarrow{\Omega}})$.

5. Suppose that i's knowledge satisfies Non-empty λ-Conjunction in the given category. Let $\mathcal{E}$ be such that $\mathcal{E} \subseteq \mathcal{L}$ and $0 < |\mathcal{E}| < \kappa$. If $\omega \in \bigcap_{e \in \mathcal{E}} K_i([e]_{\overrightarrow{\Omega}}) =$

$[\bigwedge_{e\in\mathcal{E}} k_i(e)]_{\overrightarrow{\Omega}}$ then $(1, \bigwedge_{e\in\mathcal{E}} k_i(e)) \in \omega$. Now, since $(1, (\bigwedge_{e\in\mathcal{E}} k_i(e) \to k_i(\bigwedge \mathcal{E})) \in \omega$, it follows that $(1, k_i(\bigwedge \mathcal{E})) \in \omega$, i.e., $\omega \in [k_i(\bigwedge \mathcal{E})]_{\overrightarrow{\Omega}} = K_i(\bigcap_{e\in\mathcal{E}} [e]_{\overrightarrow{\Omega}})$.

6. Consider Truth Axiom of i's knowledge. If $\omega \in K_i([e]_{\overrightarrow{\Omega}}) = [k_i(e)]_{\overrightarrow{\Omega}}$ then $(1, k_i(e)) \in \omega$. Now, since $(1, (k_i(e) \to e) \in \omega$, it follows that $(1, e) \in \omega$, i.e., $\omega \in [e]_{\overrightarrow{\Omega}}$.

7. Suppose that i's knowledge satisfies Positive Introspection in the given category. If $\omega \in K_i([e]_{\overrightarrow{\Omega}}) = [k_i(e)]_{\overrightarrow{\Omega}}$ then $(1, k_i(e)) \in \omega$. Now, since $(1, (k_i(e) \to k_i k_i(e))) \in \omega$, it follows that $(1, k_i k_i(e)) \in \omega$, i.e., $\omega \in [k_i k_i(e)]_{\overrightarrow{\Omega}} = K_i K_i([e]_{\overrightarrow{\Omega}})$.

8. Consider Negative Introspection of i's knowledge. If $\omega \in (\neg K_i)([e]_{\overrightarrow{\Omega}}) = [(\neg k_i)(e)]_{\overrightarrow{\Omega}}$ then $(1, (\neg k_i)(e)) \in \omega$. Now, since $(1, ((\neg k_i)(e) \to k_i(\neg k_i)(e))) \in \omega$, it follows that $(1, k_i(\neg k_i)(e)) \in \omega$, i.e., $\omega \in [k_i(\neg k_i)(e)]_{\overrightarrow{\Omega}} = K_i(\neg K_i)([e]_{\overrightarrow{\Omega}})$.

9. Suppose that the Kripke property is imposed on i in the given category. It follows from the assumption that $\omega \in K_i([e]_{\overrightarrow{\Omega}})$ for any $(\omega, [e]_{\overrightarrow{\Omega}}) \in \Omega \times \mathcal{D}$ such that $\omega \in \bigcap\{[f]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid \omega \in K_i([f]_{\overrightarrow{\Omega}})\}$.

Step 2.4. The above arguments establish that $\overrightarrow{\Omega} := \langle(\Omega, \mathcal{D}), (K_i)_{i\in I}, \Theta\rangle$ is a knowledge space of the given category. Finally, we show that the description map $D : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ is an inclusion map so that $\Omega \subseteq \Omega^*$.

To that end, we first establish by induction that $[\cdot]_{\overrightarrow{\Omega}} : \mathcal{L} \to \mathcal{D}$, viewed as a mapping, coincides with the semantic interpretation function $[\![\cdot]\!]_{\overrightarrow{\Omega}}$. Fix $E \in \mathcal{A}_S$. We have $\omega \in [\![E]\!]_{\overrightarrow{\Omega}} = \Theta^{-1}(E)$ iff $\Theta(\omega) \in E$ iff $\omega \in [E]_{\overrightarrow{\Omega}}$.

Next, suppose that $[\![e]\!]_{\overrightarrow{\Omega}} = [e]_{\overrightarrow{\Omega}}$ for some e. Then,

$$\omega \in [\![\neg e]\!]_{\overrightarrow{\Omega}} = \neg[\![e]\!]_{\overrightarrow{\Omega}} \text{ iff } \omega \notin [\![e]\!]_{\overrightarrow{\Omega}} \text{ iff } \omega \notin [e]_{\overrightarrow{\Omega}} \text{ iff } \omega \in [\neg e]_{\overrightarrow{\Omega}}.$$

Next, suppose that $[\![e]\!]_{\overrightarrow{\Omega}} = [e]_{\overrightarrow{\Omega}}$ for all $e \in \mathcal{E}$ with $\mathcal{E} \subseteq \mathcal{L}$ and $(0 <)|\mathcal{E}| < \kappa$. Then,

$$\omega \in \left[\!\!\left[\bigwedge \mathcal{E}\right]\!\!\right]_{\overrightarrow{\Omega}} = \bigcap_{e\in\mathcal{E}} [\![e]\!]_{\overrightarrow{\Omega}} \text{ iff } \omega \in \bigcap_{e\in\mathcal{E}} [e]_{\overrightarrow{\Omega}} = \left[\bigwedge \mathcal{E}\right]_{\overrightarrow{\Omega}}.$$

Next, assuming the induction hypothesis, we have $[\![k_i(e)]\!]_{\overrightarrow{\Omega}} = K_i([\![e]\!]_{\overrightarrow{\Omega}}) = K_i([e]_{\overrightarrow{\Omega}}) = [k_i(e)]_{\overrightarrow{\Omega}}$. The induction is complete.

Now, we show that $D(\omega) = \omega$ for all $\omega \in \Omega$. First, we consider expressions. If $(1, e) \in D(\omega)$ then $D(\omega) \in [e]$, and thus $\omega \in D^{-1}([e]) = [\![e]\!]_{\overrightarrow{\Omega}} = [e]_{\overrightarrow{\Omega}}$. Then, $(1, e) \in \omega$. Conversely, if $(1, e) \in \omega$ then $\omega \in [e]_{\overrightarrow{\Omega}} = [\![e]\!]_{\overrightarrow{\Omega}} = D^{-1}([e])$. Thus, $D(\omega) \in [e]$ and hence $(1, e) \in D(\omega)$.

Second, if $(0, s) \in \omega$ then $s = \Theta(\omega) = \Theta^*(D(\omega))$, and thus $(0, s) \in D(\omega)$. Conversely, if $(0, s) \in D(\omega)$ then $s = \Theta^*(D(\omega)) = \Theta(\omega)$, and thus $(0, s) \in \omega$. This completes the proof of the statement that D is an inclusion map, and the proof of Theorem 2 is now complete. $\square$

A.2.3 Section 3.3

Proof of Proposition 4. Part 1. Let $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ be a knowledge morphism between κ-knowledge spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$. In a similar way to HS [38], we show by induction that $\mathcal{C}_\alpha = \varphi^{-1}(\mathcal{C}'_\alpha)$ for all α.

Let $\alpha = 0$. For any $E \in \mathcal{A}_S$, we have $(\Theta)^{-1}(E) = \varphi^{-1}((\Theta')^{-1}(E))$. Thus, $\mathcal{C}_0 = \varphi^{-1}(\mathcal{C}'_0)$. Now, suppose that $\mathcal{C}_\beta = \varphi^{-1}(\mathcal{C}'_\beta)$ for all $\beta < \alpha$. Then, $\mathcal{C}_\alpha = \varphi^{-1}(\mathcal{C}'_\alpha)$ follows because the set-algebraic operations and the knowledge operators commute with the inverse φ^{-1}. Indeed, we have

$$\begin{aligned}
\mathcal{C}_\alpha &= \mathcal{A}_\kappa\left(\{\varphi^{-1}(E') \in \mathcal{D} \mid E' \in \bigcup_{\beta<\alpha} \mathcal{C}'_\beta\} \cup \bigcup_{i\in I}\{K_i(\varphi^{-1}(E')) \in \mathcal{D} \mid E' \in \bigcup_{\beta<\alpha} \mathcal{C}'_\beta\}\right)\\
&= \mathcal{A}_\kappa\left(\{\varphi^{-1}(E') \in \mathcal{D} \mid E' \in \bigcup_{\beta<\alpha} \mathcal{C}'_\beta\} \cup \bigcup_{i\in I}\{\varphi^{-1}(K'_i(E')) \in \mathcal{D} \mid E' \in \bigcup_{\beta<\alpha} \mathcal{C}'_\beta\}\right)\\
&= \varphi^{-1}\left(\mathcal{A}_\kappa\left(\{E' \in \mathcal{D} \mid E' \in \bigcup_{\beta<\alpha} \mathcal{C}'_\beta\} \cup \bigcup_{i\in I}\{K'_i(E') \in \mathcal{D} \mid E' \in \bigcup_{\beta<\alpha} \mathcal{C}'_\beta\}\right)\right) = \varphi^{-1}(\mathcal{C}'_\alpha).
\end{aligned}$$

Thus, if $\mathcal{C}'_\alpha = \mathcal{C}'_{\alpha+1}$, then $\mathcal{C}_\alpha = \mathcal{C}_{\alpha+1}$. In other words, if the κ-rank of $\overrightarrow{\Omega'}$ is α then that of $\overrightarrow{\Omega}$ is at most α.

Part 2. Fix a κ-knowledge space $\overrightarrow{\Omega}$. We show that $\mathcal{D}_\alpha = \mathcal{C}_\alpha$ for all $\alpha \leq \kappa$, where $\mathcal{D}_\alpha$ is defined in the main text. For $\alpha = 0$, it is clear that $\mathcal{D}_0 = \{\Theta^{-1}(E) \in \mathcal{D} \mid E \in \mathcal{A}_S\} = \mathcal{C}_0$. Next, suppose that $\mathcal{D}_\beta = \mathcal{C}_\beta$ for all $\beta < \alpha$. Then, it can be seen that

$$\mathcal{D}_\alpha = \mathcal{A}_\kappa\left(\left(\bigcup_{\beta<\alpha} \mathcal{D}_\beta\right) \cup \bigcup_{i\in I}\{K_i([\![e]\!]) \in \mathcal{D} \mid [\![e]\!] \in \bigcup_{\beta<\alpha} \mathcal{D}_\beta\}\right) = \mathcal{C}_\alpha.$$

Now, we have $\mathcal{C}_\kappa = \mathcal{D}_\kappa = \{[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid e \in \mathcal{L}\}$. Then, we obtain $\mathcal{C}_\kappa = \mathcal{C}_{\kappa+1}$, and the κ-rank of $\overrightarrow{\Omega}$ is at most κ. $\square$

A.3 Section 4

A.3.1 Section 4.1

Proof of Proposition 5. Part 1. First, since $\emptyset \subseteq K_i(\emptyset)$, it is immediate that $\emptyset \in \mathcal{J}_{K_i}$.

Second, we show that $K_i(E) = \max\{F \in \mathcal{D} \mid F \in \mathcal{J}_{K_i} \text{ and } F \subseteq E\} \in \mathcal{J}_{K_i}$ for each $E \in \mathcal{D}$ if K_i satisfies Truth Axiom, Monotonicity and Positive Introspection. Once this equality is established, $\mathcal{J}_{K_i}$ clearly satisfies the maximality property.

Fix $E \in \mathcal{D}$. For any $F \in \mathcal{J}_{K_i}$ with $F \subseteq E$, we have $F \subseteq K_i(F) \subseteq K_i(E)$, where the second set inclusion follows from Monotonicity. On the other hand, Positive

Introspection and Truth Axiom imply $K_i(E) \in \{F \in \mathcal{J}_{K_i} \mid F \subseteq E\}$, yielding the desired equality $K_i(E) = \max\{F \in \mathcal{J}_{K_i} \mid F \subseteq E\}$.

Third, assume Non-empty λ-Conjunction of K_i. Let $\mathcal{E}$ be a non-empty subset of $\mathcal{J}_{K_i}$ with $|\mathcal{E}| < \lambda$. We have $\bigcap \mathcal{E} \subseteq \bigcap_{E \in \mathcal{E}} K_i(E) \subseteq K_i(\bigcap \mathcal{E})$, implying $\bigcap \mathcal{E} \in \mathcal{J}_{K_i}$.

Fourth, it is clear that Necessitation implies $\Omega \in \mathcal{J}_{K_i}$, since $K_i(\Omega) = \Omega$.

Fifth, we show that Negative Introspection, together with Truth Axiom, imply that $\mathcal{J}_{K_i}$ is closed under complementation. If $E \in \mathcal{J}_{K_i}$ then $E = K_i(E)$, which implies $E^c = (\neg K_i)(E)$. It follows from Negative Introspection that $E^c = (\neg K_i)(E) \subseteq K_i(\neg K_i)(E) = K_i(E^c)$, from which we obtain $E^c \in \mathcal{J}_{K_i}$.

Part 2. Let $\mathcal{J}_i$ be a sub-collection of $\mathcal{D}$ satisfying the maximality property. First, we show that $K_{\mathcal{J}_i}$ satisfies Truth Axiom, Positive Introspection, and Monotonicity. We obtain Truth Axiom of $K_{\mathcal{J}_i}$ because $K_{\mathcal{J}_i}(E) = \max\{F \in \mathcal{J}_i \mid F \subseteq E\} \subseteq E$ for any $E \in \mathcal{D}$. Next, since $K_{\mathcal{J}_i}(E) \in \mathcal{J}_i$, we have $K_{\mathcal{J}_i}(E) \subseteq \max\{F \in \mathcal{J}_i \mid F \subseteq K_{\mathcal{J}_i}(E)\} = K_{\mathcal{J}_i} K_{\mathcal{J}_i}(E)$, establishing Positive Introspection. Finally, if $E \subseteq F$ then $K_{\mathcal{J}_i}(E)(\subseteq E) \subseteq F$. Since $K_{\mathcal{J}_i}(E) \in \{F' \in \mathcal{J}_i \mid F' \subseteq F\}$, we get $K_{\mathcal{J}_i}(E) \subseteq \max\{F' \in \mathcal{J}_i \mid F' \subseteq F\} = K_{\mathcal{J}_i}(F)$, i.e., Monotonicity is established.

Second, assume that $\mathcal{J}_i$ is closed under non-empty λ-intersection. Let $\mathcal{E}$ be a non-empty subset of $\mathcal{D}$ with $|\mathcal{E}| < \lambda$. Note that by the previous argument, $K_{\mathcal{J}_i}$ satisfies Truth Axiom and Monotonicity. It follows from Truth Axiom that $\bigcap_{E \in \mathcal{E}} K_{\mathcal{J}_i}(E) \subseteq \bigcap \mathcal{E}$. Now, $\mathcal{J}_i$ being closed under non-empty λ-intersection, we get $\bigcap_{E \in \mathcal{E}} K_{\mathcal{J}_i}(E) \in \mathcal{J}_i$, and thus $\bigcap_{E \in \mathcal{E}} K_{\mathcal{J}_i}(E) \subseteq K_{\mathcal{J}_i}(\bigcap_{E \in \mathcal{E}} K_{\mathcal{J}_i}(E)) \subseteq K_{\mathcal{J}_i}(\bigcap \mathcal{E})$, where the second set inclusion follows from Monotonicity. Hence, Non-empty λ-Conjunction is established.

Third, if $\Omega \in \mathcal{J}_i$, then $\Omega = \max\{E \in \mathcal{J}_i \mid E \subseteq \Omega\} = K_{\mathcal{J}_i}(\Omega)$, establishing Necessitation.

Fourth, assume that $\mathcal{J}_i$ is closed under complementation. Since $K_{\mathcal{J}_i}(E) \in \mathcal{J}_i$, we have $(\neg K_{\mathcal{J}_i})(E) \in \mathcal{J}_i$ and thus $(\neg K_{\mathcal{J}_i})(E) \subseteq \max\{F \in \mathcal{J}_i \mid F \subseteq (\neg K_{\mathcal{J}_i})(E)\} = K_{\mathcal{J}_i}(\neg K_{\mathcal{J}_i})(E)$.

Part 3. First, $K_i = K_{\mathcal{J}_{K_i}}$ simply follows from Equation (3). Next, we show that $\mathcal{J}_i = \mathcal{J}_{K_{\mathcal{J}_i}}$. If $E \in \mathcal{J}_i$, then we obtain $E \subseteq \max\{F \in \mathcal{J}_i \mid F \subseteq E\} = K_{\mathcal{J}_i}(E)$ and thus $E \in \mathcal{J}_{K_{\mathcal{J}_i}}$, establishing $\mathcal{J}_i \subseteq \mathcal{J}_{K_{\mathcal{J}_i}}$. Conversely, if $E \in \mathcal{J}_{K_{\mathcal{J}_i}}$, then we have $E = K_{\mathcal{J}_i}(E) = \max\{F \in \mathcal{J}_i \mid F \subseteq E\} \in \mathcal{J}_i$ by the maximality property of $\mathcal{J}_i$. Thus, we have $\mathcal{J}_{K_{\mathcal{J}_i}} \subseteq \mathcal{J}_i$.

Part 4. Let κ be an infinite cardinal or $\kappa = \infty$. We show that the maximality property implies that $\mathcal{J}_{K_i}$ is closed under κ-union, where ∞-union refers to arbitrary union. Recall that the maximality property, by definition, implies $\emptyset \in \mathcal{J}_{K_i}$. Thus, $\mathcal{J}_{K_i}$ is closed under the empty union. Now, let $\mathcal{E}$ be any non-empty subset of $\mathcal{J}_{K_i}$ with $|\mathcal{E}| < \kappa$. For any $E \in \mathcal{E}$, we have $E \subseteq K_i(E) \subseteq K_i(\bigcup \mathcal{E})$, and thus $\bigcup \mathcal{E} \subseteq K_i(\bigcup \mathcal{E})$, establishing $\bigcup \mathcal{E} \in \mathcal{J}_{K_i}$. Indeed, we have shown that, for any $\mathcal{E} \subseteq \mathcal{J}_{K_i}$, if $\bigcup \mathcal{E} \in \mathcal{D}$ then $\bigcup \mathcal{E} \in \mathcal{J}_{K_i}$.

We establish the converse when $\kappa = \infty$. If $\mathcal{J}_{K_i}$ is closed under arbitrary union, then, for each $E \in \mathcal{D}$, we have

$$\max\{F \in \mathcal{D} \mid F \in \mathcal{J}_{K_i} \text{ and } F \subseteq E\} = \bigcup\{F \in \mathcal{D} \mid F \in \mathcal{J}_{K_i} \text{ and } F \subseteq E\} \in \mathcal{J}_{K_i}.$$

Thus, $\mathcal{J}_{K_i}$ satisfies the maximality property. □

Proof of Corollary 2. Note that Truth Axiom and Negative Introspection imply Positive Introspection (again, see, for example, Aumann [5, p. 270]). By Proposition 5, the knowledge operator K_i satisfies κ-Conjunction (including Necessitation) iff $\mathcal{J}_{K_i}$ is closed under κ-intersection. Now, it follows again from Proposition 5 that $\mathcal{J}_{K_i}$ is closed under κ-union and complementation. Thus, it is closed under κ-intersection. □

Remark A.4 (Remark on Proposition 5: Counterexamples without Truth Axiom, Monotonicity, or Positive Introspection)**.** Throughout the following three examples, let $\Omega = \{\omega_1, \omega_2\}$ and $\mathcal{D} = \mathcal{P}(\Omega)$. First, we define K_i and K_i' as follows: $K_i(\emptyset) = \emptyset$, $K_i(E) = \{\omega_1\}$ for any $E \in \mathcal{P}(\Omega) \setminus \{\emptyset\}$; and $K_i'(\emptyset) = K_i'(\{\omega_2\}) = \emptyset$, $K_i'(\{\omega_1\}) = K_i'(\Omega) = \{\omega_1\}$. The operator K_i' violates Truth Axiom. It is clear that $K_i \neq K_i'$ but $\mathcal{J}_{K_i} = \mathcal{J}_{K_i'}$. As an additional remark, while any operator K_i satisfying Positive Introspection and Monotonicity also trivially satisfies the equality $K_i(E) = \max\{F \in \mathcal{D} \mid F \in \mathcal{J}_{K_i} \text{ and } F \subseteq K_i(E)\}$, there may be multiple operators which have the same self-evident collections due to the lack of Truth Axiom.

Second, let K_i and K_i' be defined as follows: $K_i(\{\omega_1\}) = K_i(\Omega) = \{\omega_1\}$, $K_i(E) = \emptyset$ for any $E \in \{\emptyset, \{\omega_2\}\}$; and $K_i'(\{\omega_1\}) = \{\omega_1\}$ and $K_i'(E) = \emptyset$ for any $E \in \mathcal{D} \setminus \{\{\omega_1\}\}$. The operator K_i' violates Monotonicity. It is clear that $K_i \neq K_i'$ but $\mathcal{J}_{K_i} = \mathcal{J}_{K_i'}$.

Third, we define K_i and K_i' as follows: $K_i(E) = \emptyset$ for any $E \in \mathcal{D}$; and $K_i'(\Omega) = \{\omega_1\}$ and $K_i'(E) = \emptyset$ for any $E \in \mathcal{D} \setminus \{\Omega\}$. The operator K_i' violates Positive Introspection. It is clear that $K_i \neq K_i'$ but $\mathcal{J}_{K_i} = \mathcal{J}_{K_i'}$.

Proof of Proposition 6.

1. First, it is clear that $\mathcal{D}' \subseteq \mathcal{J}(\mathcal{D}')$. Second, we show that $\mathcal{J}(\mathcal{D}')$ satisfies the maximality property. It is clear that $\emptyset \in \mathcal{J}(\mathcal{D}')$. Fix $E \in \mathcal{D}$. It follows from the definition that

$$\begin{aligned}&\{F \in \mathcal{D} \mid F \in \mathcal{J}(\mathcal{D}') \text{ and } F \subseteq E\}\\ &= \{F \in \mathcal{D} \mid \text{if } \omega \in F \text{ then there is } F' \in \mathcal{D}' \text{ with } \omega \in F' \subseteq F \subseteq E\}.\end{aligned}$$

Then, it can be seen that

$$\begin{aligned}&\max\{F \in \mathcal{D} \mid F \in \mathcal{J}(\mathcal{D}') \text{ and } F \subseteq E\}\\ &= \{\omega \in \Omega \mid \text{there is } F' \in \mathcal{D}' \text{ with } \omega \in F' \subseteq E\} \in \mathcal{J}(\mathcal{D}').\end{aligned}$$

Third, take any collection $\mathcal{J}'$ containing $\mathcal{D}'$ and satisfying the maximality property. Take any $E \in \mathcal{J}(\mathcal{D}')$. Then, for any $\omega \in E$, there is $F \in \mathcal{D}'$ with $\omega \in F \subseteq E$. Since $\mathcal{D}' \subseteq \mathcal{J}'$, we have $F \in \mathcal{J}'$. Now, we obtain

$$E = \{\omega \in \Omega \mid \text{there is } F \in \mathcal{J}' \text{ such that } \omega \in F \subseteq E\} = \max\{F \in \mathcal{J}' \mid F \subseteq E\} \in \mathcal{J}'.$$

Thus, we get $\mathcal{J}(\mathcal{D}') \subseteq \mathcal{J}'$.

2. First, we have $\mathcal{J}(\mathcal{D}') \cup \{\Omega\} \subseteq \mathcal{J}(\mathcal{D}' \cup \{\Omega\})$. Second, it can be easily seen that $\mathcal{J}(\mathcal{D}') \cup \{\Omega\}$ satisfies the maximality property and contains $\mathcal{D}' \cup \{\Omega\}$, leading to $\mathcal{J}(\mathcal{D}' \cup \{\Omega\}) \subseteq \mathcal{J}(\mathcal{D}') \cup \{\Omega\}$.

3. It is enough to prove the second assertion. By construction, $\mathcal{J}_{\lambda\text{-Con}}$ contains $\mathcal{D}$, is closed under non-empty λ-intersection, and satisfies the maximality property. For any collection $\mathcal{J}'$ containing $\mathcal{D}'$ and satisfying the maximality property and non-empty λ-Conjunction, we have

$$\{E \in \mathcal{D} \mid E = \bigcap \mathcal{E} \text{ for some } \mathcal{E} \subseteq \mathcal{D}' \text{ with } 0 < |\mathcal{E}| < \lambda\} \subseteq \mathcal{J}',$$

establishing $\mathcal{J}_{\lambda\text{-Con}}(\mathcal{D}') \subseteq \mathcal{J}'$.

4. The following counterexample shows that $\mathcal{J}(\mathcal{D}')$ is not necessarily closed under complementation even if $\mathcal{D}'$ is. Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$. If $\mathcal{D}' = \{\emptyset, \{\omega_1\}, \{\omega_3\}, \{w_1, \omega_2\}, \{\omega_2, \omega_3\}, \Omega\}$ then $\mathcal{J}(\mathcal{D}') = \mathcal{P}(\Omega) \setminus \{\{\omega_2\}\}$. While $\mathcal{D}'$ is closed under complementation, $\mathcal{J}(\mathcal{D}')$ is not.

   Let α be an ordinal which satisfies the condition specified in the statement of the proposition. We show that $\mathcal{C}_\alpha$ is the desired collection. First, it is clear that $\mathcal{D}' \subseteq \mathcal{C}_\alpha$. Second, if $E \in \mathcal{C}_\alpha$, then $E^c \in \mathcal{C}_{\alpha+1} = \mathcal{C}_\alpha$. Thus, $\mathcal{C}_\alpha$ is closed under complementation.

   Third, we show that $\mathcal{C}_\alpha$ satisfies the maximality property. For any $E \in \mathcal{D}$, we have

$$\{\omega \in \Omega \mid \text{if } \omega \in E \text{ then there is } F \in \mathcal{C}_\alpha \text{ such that } \omega \in F \subseteq E\} \in \mathcal{J}(\mathcal{C}_\alpha) = \mathcal{C}_\alpha,$$

   where the last equality follows because $\mathcal{C}_\alpha \subseteq \mathcal{J}(\mathcal{C}_\alpha) \subseteq \mathcal{C}_{\alpha+1} = \mathcal{C}_\alpha$. Note that the above set is the largest event in $\mathcal{C}_\alpha$ which is contained in E. Thus, $\mathcal{C}_\alpha$ satisfies the maximality property.

   Fourth, let $\mathcal{J}'$ be a collection with the following properties: (i) $\mathcal{D}' \subseteq \mathcal{J}'$; (ii) $\mathcal{J}'$ is closed under complementation; and (iii) $\mathcal{J}'$ satisfies the maximality property. We show that $\mathcal{C}_\alpha \subseteq \mathcal{J}'$. By assumption, $\mathcal{C}_0 \subseteq \mathcal{J}'$. If $\mathcal{C}_\beta \subseteq \mathcal{J}'$, then it is clear that $\mathcal{C}_{\beta+1} \subseteq \mathcal{J}'$. If $\mathcal{C}_\beta \subseteq \mathcal{J}'$ for all $\beta < \gamma$, then it is clear that $\bigcup_{\beta<\gamma} \mathcal{C}_\beta \subseteq \mathcal{J}'$. Hence, $\mathcal{C}_\alpha \subseteq \mathcal{J}'$.

5. It follows from Proposition 5 that the maximality property is characterized as the closure under arbitrary union. Then, the assertions clearly hold.

□

Proof of Corollary 3. The first condition clearly implies the second. Conversely, if $F \in \mathcal{J}(\mathcal{E})$ and $\omega \in F$ then it follows from Equation (4) that there is $E \in \mathcal{E}$ with $\omega \in E \subseteq F$. Now, by the second condition, there is $E' \in \mathcal{E}'$ such that $\omega \in E' \subseteq E \subseteq F$. Since $\omega \in F$ is arbitrary, we have $F \in \mathcal{J}(\mathcal{E}')$. □

Finally, since a knowledge space can be equivalently defined by players' self-evident collections instead of their knowledge operators, we express the condition for a mapping to preserve players' knowledge (i.e., Condition (2) in Definition 3) solely in terms of players' self-evident collections when players' knowledge satisfy Truth Axiom, Positive Introspection, and Monotonicity.

Proposition A.3 (Characterzing Knowledge Morphisms by Self-Evident Collections)**.** *Let $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$ be knowledge spaces such that player i's knowledge operator satisfies Truth Axiom, Positive Introspection, and Monotonicity. Let $\varphi : \Omega \to \Omega'$ be a mapping. Condition (2) in Definition 3 can be equivalently stated as the following two conditions for player i.*

1. *If $E' \in \mathcal{J}'_{K'_i}$, then $\varphi^{-1}(E') \in \mathcal{J}_{K_i}$.*
2. *For any $F \in \mathcal{J}_{K_i}$ and $E' \in \mathcal{D}'$ with $\varphi(F) \subseteq E'$, there is $F' \in \mathcal{J}'_{K'_i}$ such that $F' \subseteq E'$ and $\varphi(F) \subseteq F'$.*

By slightly abusing the notion of continuity in topology, we can regard the first condition as the regularity condition that a knowledge morphism "continuously" maps each player's knowledge from one space to another. On the other hand, if a self-evident event $E \in \mathcal{J}_{K_i}$ satisfies $\varphi(E) \in \mathcal{D}'$, then it is indeed self-evident, i.e., $\varphi(E) \in \mathcal{J}'_{K'_i}$. A knowledge morphism, however, does not necessarily map an event to an event, and this latter condition captures the sense in which a knowledge morphism preserves self-evident events with the presence of domain restrictions.

Proof of Proposition A.3. Suppose Condition (2) in Definition 3. First, if $E' \in \mathcal{J}'_{K'_i}$, then $E' \subseteq K'_i(E')$. Now, we have $\varphi^{-1}(E') \subseteq \varphi^{-1}(K'_i(E')) = K_i(\varphi^{-1}(E'))$, implying that $\varphi^{-1}(E') \in \mathcal{J}_{K_i}$. Second, let $F \in \mathcal{J}_{K_i}$ and $E' \in \mathcal{D}'$ with $\varphi(F) \subseteq E'$. We have $F \subseteq \varphi^{-1}(\varphi(F)) \subseteq \varphi^{-1}(E')$. Then, $F \subseteq K_i(F) \subseteq K_i(\varphi^{-1}(E')) = \varphi^{-1}(K'_i(E'))$. Let $F' := K'_i(E') \in \mathcal{J}'_{K'_i}$. We have $F' \subseteq E'$ and $\varphi(F) \subseteq \varphi(\varphi^{-1}(F')) \subseteq F'$. Conversely, suppose the two conditions in the statement. Take any $E' \in \mathcal{D}'$. We have $K'_i(E') \in \mathcal{J}'_{K'_i}$. It follows from the first condition that $\varphi^{-1}(K'_i(E')) \in \mathcal{J}_{K_i}$, and hence $\varphi^{-1}(K'_i(E')) \subseteq K_i(\varphi^{-1}(K'_i(E'))) \subseteq K_i(\varphi^{-1}(E'))$. Next, we have $\varphi^{-1}(E') \in \mathcal{D}$, $F := K_i(\varphi^{-1}(E')) \in \mathcal{J}_{K_i}$, and $\varphi(K_i(\varphi^{-1}(E'))) \subseteq \varphi(\varphi^{-1}(E')) \subseteq E'$. Now, the second condition implies that there exists $F' \in \mathcal{J}_{K'_i}$ such that $\varphi(K_i(\varphi^{-1}(E'))) = \varphi(F) \subseteq F' \subseteq K'_i(E')$. Thus, we obtain $K_i(\varphi^{-1}(E')) \subseteq \varphi^{-1}(K'_i(E'))$. The proof is complete. □

A.3.2 Section 4.2

Proof of Proposition 7. 1. Fix $E \in \mathcal{D}$. If $\omega \in C(E)$, then there is $F \in \mathcal{J}_I$ such that $\omega \in F \subseteq C(E)$. Since F is a common basis, we have $F \subseteq K_I(C(E))$. This implies that $\omega \in F \subseteq C(C(E))$, establishing Positive Introspection of C.

2. First, suppose that there is $i \in I$ such that K_i satisfies No-Contradiction Axiom. Then, $\emptyset \subseteq C(\emptyset) \subseteq K_I(\emptyset) \subseteq K_i(\emptyset) = \emptyset$. Second, if K_i satisfies Consistency for some $i \in I$, then $C(E) \cap C(E^c) \subseteq K_I(E) \cap K_I(E^c) \subseteq K_i(E) \cap K_i(E^c) = \emptyset$. Third, if K_i satisfies Truth Axiom for some $i \in I$, then $C(E) \subseteq K_I(E) \subseteq K_i(E) \subseteq E$.

3. Suppose that each K_i satisfies Monotonicity. Fix $E, F \in \mathcal{D}$ with $E \subseteq F$. If $\omega \in C(E)$ then there is $E' \in \mathcal{J}_I$ with $\omega \in E' \subseteq K_I(E) \subseteq K_I(F)$. Then, we get $\omega \in C(F)$, establishing Monotonicity of C.

4. Assume that each K_i satisfies Necessitation. Then, we have $\Omega \in \mathcal{J}_I$ and $K_I(\Omega) = \Omega$, leading to $C(\Omega) = \Omega$. Conversely, if C satisfies Necessitation then $\Omega \subseteq C(\Omega) \subseteq K_I(\Omega) \subseteq K_i(\Omega) \subseteq \Omega$ for all $i \in I$.

5. Assume that each K_i satisfies Non-empty λ-Conjunction as well as Monotonicity. Given Monotonicity, we have $\mathcal{J}_I = \bigcap_{i \in I} \mathcal{J}_{K_i}$. Now, since each $\mathcal{J}_{K_i}$ is closed under non-empty λ-intersection, so is $\mathcal{J}_I$. Then, it follows from Proposition 6 that $\mathcal{J}(\mathcal{J}_I)$ is also closed under non-empty λ-intersection. Let $\mathcal{E}$ be a non-empty set of events with $0 < |\mathcal{E}| < \lambda$ such that $\omega \in \bigcap_{E \in \mathcal{E}} C(E)$. For each $E \in \mathcal{E}$, there is $F_E \in \mathcal{J}_I$ such that $\omega \in F_E \subseteq K_I(E)$. Then, we obtain $\omega \in \bigcap_{E \in \mathcal{E}} F_E \subseteq \bigcap_{E \in \mathcal{E}} \bigcap_{i \in I} K_i(E) = \bigcap_{i \in I} \bigcap_{E \in \mathcal{E}} K_i(E) \subseteq K_I(\bigcap \mathcal{E})$. Since $\bigcap_{E \in \mathcal{E}} F_E \in \mathcal{J}_I$, it follows that $\omega \in C(\bigcap \mathcal{E})$. Hence, C satisfies Non-empty λ-Conjunction.

   Next, we provide a counterexample when Monotonicity is violated. Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$. Let K_i be defined as in the table below for each $i \in I = \{1, 2\}$.

E	$K_1(E)$	$K_2(E)$	$K_I(E)$	$C(E)$	$CC(E)$	$(\neg C)(E)$	$C(\neg C)(E)$
$\emptyset$	Ω	Ω	Ω	Ω	Ω	$\emptyset$	Ω
$\{\omega_1\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_3\}$	$\emptyset$
$\{\omega_2\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_1\}$	$\{\omega_1, \omega_2\}$
$\{\omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_3\}$	$\emptyset$	Ω	Ω	Ω
$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_3\}$	$\emptyset$
$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_3\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2, \omega_3\}$	Ω
$\{\omega_2, \omega_3\}$	Ω	Ω	Ω	Ω	Ω	$\emptyset$	Ω
Ω	Ω	Ω	Ω	Ω	Ω	$\emptyset$	Ω

   Clearly, we have $\mathcal{J}_1 = \mathcal{D} \setminus \{\{\omega_1, \omega_3\}\}$ and $\mathcal{J}_2 = \mathcal{D}$. On the other hand, we have $\mathcal{J}_I = \mathcal{D} \setminus \{\{\omega_3\}, \{\omega_1, \omega_3\}\}$. Then, the common knowledge operator C is depicted as in the table. C violates (Non-empty) Finite Conjunction because $C(\{\omega_1, \omega_3\}) \cap C(\{\omega_2, \omega_3\}) = \{\omega_1\} \not\subseteq \emptyset = C(\{\omega_3\})$. Incidentally, C violates all the properties except for Positive Introspection and Necessitation.

6. The above example is indeed an example in which C violates Negative Introspection while individual knowledge operators satisfy Negative Introspection (see the table below).

E	$K_1(E)$	$K_2(E)$	$(\neg K_1)(E)$	$(\neg K_2)(E)$	$K_1(\neg K_1)(E)$	$K_2(\neg K_2)(E)$
$\emptyset$	Ω	Ω	$\emptyset$	$\emptyset$	Ω	Ω
$\{\omega_1\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_2\}$	$\{\omega_3\}$	$\{\omega_3\}$	$\{\omega_2,\omega_3\}$	$\{\omega_1,\omega_3\}$
$\{\omega_2\}$	$\{\omega_2,\omega_3\}$	$\{\omega_2,\omega_3\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_2\}$
$\{\omega_3\}$	$\{\omega_2,\omega_3\}$	$\{\omega_1,\omega_3\}$	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_1,\omega_2\}$	$\{\omega_2,\omega_3\}$
$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_2\}$	$\{\omega_3\}$	$\{\omega_3\}$	$\{\omega_2,\omega_3\}$	$\{\omega_1,\omega_3\}$
$\{\omega_1,\omega_3\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_3\}$	$\{\omega_3\}$	$\{\omega_2\}$	$\{\omega_2,\omega_3\}$	$\{\omega_2,\omega_3\}$
$\{\omega_2,\omega_3\}$	Ω	Ω	$\emptyset$	$\emptyset$	Ω	Ω
Ω	Ω	Ω	$\emptyset$	$\emptyset$	Ω	Ω

7. Assume that $(\Omega,\mathcal{D})$ is a complete algebra and that each player's knowledge satisfies Truth Axiom, Positive Introspection, and Monotonicity. First, it is clear that $\mathcal{J}(\mathcal{J}_I)=\bigcap_{i\in I}\mathcal{J}_{K_i}$, as the maximality property is characterized by the closure under arbitrary union (Proposition 5). In this case, $\bigcap_{i\in I}\mathcal{J}_{K_i}$ also inherits Negative Introspection (as well as λ-Conjunction) as they are characterized by the closure under complementation (λ-intersection).

8. The first assertion follows from the previous arguments. Next, fix $E\in\mathcal{D}$. Observe that we have

$$K(E)=\{\omega\in\Omega \mid \text{there is } F\in\mathcal{J}_K \text{ such that } \omega\in F\subseteq K(E)\}.$$

Observe also that if $F\in\mathcal{J}_K$ then $F\subseteq K(F)\subseteq K_I(F)$. Thus, $\mathcal{J}_K\subseteq\bigcap_{i\in I}\mathcal{J}_{K_i}=\mathcal{J}_I$, where the last equality follows from Monotonicity. Then, we can see that

$$K(E)\subseteq\{\omega\in\Omega \mid \text{there is } F\in\mathcal{J}_I \text{ such that } \omega\in F\subseteq K_I(E)\}=C(E).$$

9. Note that $K_I^\alpha(E)$ is an event of a given κ-knowledge space for any $E\in\mathcal{D}$ and for any $\alpha<\kappa$. By definition, $C(\cdot)\subseteq K_I^1(\cdot)$. Suppose $C(\cdot)\subseteq K_I^\beta(\cdot)$. Now, if $\omega\in C(E)$, then there is $F\in\mathcal{J}_I$ such that $\omega\in F\subseteq C(E)\subseteq K_I^\beta(E)$. Since F is a common basis, we have $\omega\in F\subseteq K_I^{\beta+1}(E)$. Next, if $C(E)\subseteq K_I^\beta(E)$ for all $\beta<\alpha$, then it is clear that $C(E)\subseteq\bigcap_{\beta:1\leq\beta<\alpha}K_I^\beta(E)=K_I^\alpha(E)$.

□

Remark A.5 (Common Knowledge among $\{i\}$ need not be i's Knowledge without Monotonicity or Positive Introspection)**.** Let $(\Omega,\mathcal{D})=(\{\omega_1,\omega_2\},\mathcal{P}(\Omega))$. Consider the knowledge operators K_1 and K_2 as depicted in the following table. While K_1 violates Positive Introspection, K_2 does Monotonicity. In either case, $K_i\neq C_{\{i\}}$.

E	$K_1(E)$	$K_1K_1(E)$	$C_{\{1\}}(E)$	E	$K_2(E)$	$K_2K_2(E)$	$C_{\{2\}}(E)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\{\omega_1\}$	$\{\omega_1\}$	$\emptyset$
$\{\omega_1\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\emptyset$
$\{\omega_2\}$	$\{\omega_1\}$	$\emptyset$	$\emptyset$	$\{\omega_2\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\emptyset$
Ω	$\{\omega_1\}$	$\emptyset$	$\emptyset$	Ω	$\emptyset$	$\{\omega_1\}$	$\emptyset$

Proof of Proposition 8. Given that each K_i satisfies Finite Conjunction, the operator K_I satisfies Finite Conjunction as well. That is, $K_I(E \cap F) = K_I(E) \cap K_I(F)$ for any $E, F \in \mathcal{D}$. Now, we show that if each K_i satisfies Monotonicity then

$$C(E) = \max\{F \in \mathcal{D} \mid F \subseteq K_I(E) \cap K_I(F)\}. \tag{A.1}$$

First, by definition, $C(E) \subseteq K_I(E)$. Second, if $\omega \in C(E)$, then there is $F \in \mathcal{J}_I$ such that $\omega \in F \subseteq C(E)$. Now, since F is a common basis, we have $\omega \in F \subseteq K_I(C(E))$, establishing $C(E) \subseteq K_I(C(E))$.

Conversely, take any $F \in \mathcal{D}$ with $F \subseteq K_I(E) \cap K_I(F)$. Then, $F \in \bigcap_{i \in I} \mathcal{J}_{K_i} = \mathcal{J}_I$ (recall that the equality follows from Monotonicity) and $F \subseteq K_I(E)$. Thus, we have $F \subseteq C(E)$. Hence, we obtain the desired expression (A.1).

Finally, consider

$$\max\{F \in \mathcal{D} \mid F \subseteq K_I(E) \cap K_I(F)\} = \max\{F \in \mathcal{D} \mid F = K_I(E) \cap K_I(F)\}.$$

This is a variant of Tarski's fixed point theorem, stating that the greatest element F satisfying $F \subseteq f_E(F) := K_I(E) \cap K_I(F)$ of a monotone operator $f_E(\cdot)$, given that it exists, is indeed the greatest fixed point. Indeed, since $F \subseteq f_E(F)$, we have $f_E(F) \subseteq f_E(f_E(F))$. Since F is the greatest such point, we have $f_E(F) \subseteq F$, i.e., $F = f_E(F) = K_I(E) \cap K_I(F)$. □

Proof of Corollary 4. Here we provide an alternative proof instead of the reasoning in the main text. Fix $E \in \mathcal{D}$. First, it follows from Proposition 7 (9) that $C(E) \subseteq K_I^\infty(E)$ (without Monotonicity or Countable Conjunction). Conversely, observe first that $K^{n+1}(E) \subseteq K_i(K^n(E))$ for all $n \in \mathbb{N}$ and $i \in I$. Since each K_i satisfies Countable Conjunction, we have $\bigcap_{n \in \mathbb{N}} K_i(K^n(E)) = K_i(\bigcap_{n \in \mathbb{N}} K^n(E))$, which shows that $K_I^\infty(E) \in \mathcal{J}_i$ for each $i \in I$, that is, $K_I^\infty(E) \in \bigcap_{i \in I} \mathcal{J}_i = \mathcal{J}_I$, where the last equality follows from Monotonicity of $(K_i)_{i \in I}$. Since $K_I^\infty(E) \subseteq K_I(E)$, we obtain $K_I^\infty(E) \subseteq C(E)$. □

Proof of Proposition 9. The assertions follow from the reasoning in the main text, and thus the proof is omitted. □

Remark A.6 (Preservation of Common Knowledge in ∞-Knowledge Spaces)**.** Consider any ∞-knowledge space satisfying Monotonicity. Fix $E \in \mathcal{D}$. Consider a sequence $(K_I^\alpha(E))_\alpha$ where α is a limit ordinal. Since it is decreasing, there is a least ordinal such that $K_I^\alpha(E) = K_I^\beta(E)$ for any limit ordinal β with $\beta \geq \alpha$. Let α_E be such a limit ordinal for each $E \in \mathcal{D}$. Now, we have a limit ordinal α such that $\alpha \geq \alpha_E$ for all $E \in \mathcal{D}$. Then, we have $K_I^\alpha(E) \subseteq K_I(E)$ and $K_I^\alpha(E) \subseteq K^{\alpha+1}(E)$, from which we obtain $K_I^\alpha(E) \subseteq C(E)$. Indeed, the equality holds by Proposition 7 (9). This means that common knowledge is automatically preserved in ∞-knowledge spaces satisfying Monotonicity.

Generally, the following proposition characterizes the preservation of common knowledge.

Proposition A.4 (Preservation of Common Knowledge)**.** *Let $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ be a knowledge morphism between knowledge spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$.*

1. *The following are equivalent.*
   (a) *$\varphi^{-1}(C'(E')) \subseteq C(\varphi^{-1}(E'))$ for all $E' \in \mathcal{D}'$.*
   (b) *For any $F' \in \mathcal{J}_I'$ and $E' \in \mathcal{D}'$ with $F' \subseteq K_I'(E')$, there is $F \in \mathcal{J}(\mathcal{J}_I)$ such that $\varphi^{-1}(F') \subseteq F$ and $F \subseteq K_I(\varphi^{-1}(E'))$.*
2. *The following are equivalent.*
   (a) *$C(\varphi^{-1}(E')) \subseteq \varphi^{-1}(C'(E'))$ for all $E' \in \mathcal{D}'$.*
   (b) *For any $F \in \mathcal{J}_I$ and $E' \in \mathcal{D}'$ with $\varphi(F) \subseteq K_I'(E')$, there is $F' \in \mathcal{J}(\mathcal{J}_I')(= \mathcal{J}_{\mathcal{D}'}(\mathcal{J}_I'))$ such that $\varphi(F) \subseteq F'$ and $F' \subseteq K_I(E')$.*

In Proposition A.4, if the knowledge space $\overrightarrow{\Omega}$ satisfies Monotonicity then Condition (1b) is always satisfied. This is because we can take $F = \varphi^{-1}(F')$.

Proof of Proposition A.4. 1. Suppose that $\varphi^{-1}(C'(\cdot)) \subseteq C(\varphi^{-1}(\cdot))$. Take $F' \in \mathcal{J}_I'$ and $E' \in \mathcal{D}'$ with $F' \subseteq K_I'(E')$. Then, $F' \subseteq C'(E') \subseteq K_I'(E')$. We obtain $\varphi^{-1}(F') \subseteq \varphi^{-1}(C'(E')) \subseteq F \subseteq K_I(\varphi^{-1}(E'))$, where $F := C(\varphi^{-1}(E')) \in \mathcal{J}(\mathcal{J}_I)$.

   Conversely, suppose (1b). Take any $E' \in \mathcal{D}'$ and assume that $\omega \in \varphi^{-1}(C'(E'))$, i.e., $\varphi(\omega) \in C'(E')$. Then, there is $F' \in \mathcal{J}_I'$ such that $\varphi(\omega) \in F' \subseteq K_I'(E')$. By (1b), there is $F \in \mathcal{J}(\mathcal{J}_I)$ such that $\omega \in \varphi^{-1}(F') \subseteq F \subseteq \varphi^{-1}(K_I'(E')) = K_I(\varphi^{-1}(E'))$. Now, it follows that $\omega \in C(\varphi^{-1}(E'))$.

2. Suppose that $C(\varphi^{-1}(\cdot)) \subseteq \varphi^{-1}(C'(\cdot))$. Let $F \in \mathcal{J}_I$ and $E' \in \mathcal{D}'$ with $\varphi(F) \subseteq E'$. Then, we have $F \subseteq \varphi^{-1}(\varphi(F)) \subseteq \varphi^{-1}(E')$. Since $F \subseteq K_I(\varphi^{-1}(E'))$, we have $F \subseteq C(\varphi^{-1}(E')) \subseteq \varphi^{-1}(C'(E'))$. Let $F' := C'(E') \in \mathcal{J}'(\mathcal{J}_I')$. We have $F' \subseteq K_I'(E')$ and $\varphi(F) \subseteq \varphi(\varphi^{-1}(F')) \subseteq F'$.

   Conversely, suppose (2b). Take any $E' \in \mathcal{D}'$ and suppose that $\omega \in C(\varphi^{-1}(E'))$. Then, there is $F \in \mathcal{J}_I$ such that $\omega \in F \subseteq K_I(\varphi^{-1}(E')) = \varphi^{-1}(K_I'(E'))$. Thus, $\varphi(\omega) \in \varphi(F) \subseteq \varphi\varphi^{-1}(K_I'(E')) \subseteq K_I'(E')$. Now, there is $F' \in \mathcal{J}(\mathcal{J}_I')$ such that $\varphi(\omega) \in \varphi(F) \subseteq F' \subseteq K_I'(E')$. Then, we obtain $\varphi(\omega) \in C'(E')$, i.e., $\omega \in \varphi^{-1}(C'(E'))$.

$\square$

Proof of Proposition 10. Since Monotonicity is assumed, it follows from Proposition A.4 that we need only to prove that $D^{-1}(C^*([e])) \supseteq C(D^{-1}([e]))$ for all $[e] \in \mathcal{D}^*$. Now, it follows from the assumptions of the statement that $C^* = K_i^*$ for all $i \in I$. Then, we have $C(D^{-1}([e])) \subseteq K_i(D^{-1}([e])) = D^{-1}(C^*([e]))$ for all $[e] \in \mathcal{D}^*$. $\square$

A.4 Section 5

A.4.1 Section 5.1

Remark A.7 (Equivalence between Knowledge Operaors and Knowledge-Type Mappings)**.** In the first step, we show that t_{K_i} inherits the properties of K_i. Fix $\omega \in \Omega$.

1. t_{K_i} clearly inherits No-Contradiction Axiom: $t_{K_i}(\omega)(\emptyset) = 0$.

2. Consider Consistency. If $t_{K_i}(\omega)(E) = 1$ then $\omega \in K_i(E) \subseteq (\neg K_i)(E^c)$, and thus $t_{K_i}(\omega)(E^c) = 0$, establishing $t_{K_i}(\omega)(E) \leq 1 - t_{K_i}(\omega)(E^c)$.

3. Consider Monotonicity. Let $E, F \in \mathcal{D}$ be such that $E \subseteq F$. If $t_{K_i}(\omega)(E) = 1$ then $\omega \in K_i(E) \subseteq K_i(F)$ and thus $t_{K_i}(\omega)(F) = 1$.

4. Necessitation is immediate: $t_{K_i}(\omega)(\Omega) = 1$.

5. Consider Non-empty λ-Conjunction. Let $\mathcal{E}$ be such that $\mathcal{E} \subseteq \mathcal{D}$ and $0 < |\mathcal{E}| < \kappa$. Suppose that $t_{K_i}(\omega)(E) = 1$ for all $E \in \mathcal{E}$ (i.e., $\min_{E \in \mathcal{E}} t_{K_i}(\omega)(E) = 1$). Then, since $\omega \in \bigcap_{E \in \mathcal{E}} K_i(E) \subseteq K_i(\bigcap \mathcal{E})$, we obtain $t_{K_i}(\omega)(\bigcap \mathcal{E}) = 1$.

6. Consider Truth Axiom. If $t_{K_i}(\omega)(E) = 1$ then $\omega \in K_i(E) \subseteq E$.

7. Consider Positive Introspection. If $t_{K_i}(\omega)(E) = 1$ then $\omega \in K_i(E) \subseteq K_i K_i(E)$. Thus, $t_{K_i}(\omega)(\{\omega' \in \Omega \mid t_{K_i}(\omega')(E) = 1\}) = t_{K_i}(\omega)(K_i(E)) = 1$.

8. Consider Negative Introspection. If $t_{K_i}(\omega)(E) = 0$ then $\omega \in (\neg K_i)(E) \subseteq K_i(\neg K_i)(E)$. Thus, $t_{K_i}(\omega)(\{\omega' \in \Omega \mid t_{K_i}(\omega')(E) = 0\}) = t_{K_i}(\omega)((\neg K_i)(E)) = 1$.

9. The Kripke property follows because $b_{t_{K_i}} = b_{K_i}$.

In the second step, we show that K_{t_i} inherits the properties of t_i. Fix $\omega \in \Omega$.

1. K_{t_i} inherits No-Contradiction Axiom: $K_{t_i}(E) = \{\omega \in \Omega \mid t_i(\omega)(\emptyset) = 1\} = \emptyset$.

2. Consider Consistency. If $\omega \in K_{t_i}(E)$ then $t_i(\omega)(E) = 1 \leq 1 - t_i(\omega)(E^c)$, and thus $t_i(\omega)(E^c) = 0$. Then, $\omega \in (\neg K_{t_i})(E^c)$.

3. Consider Monotonicity. Let $E, F \in \mathcal{D}$ be such that $E \subseteq F$. If $\omega \in K_{t_i}(E)$ then $1 = t_i(\omega)(E) \leq t_i(\omega)(F)$. Thus, $\omega \in K_{t_i}(F)$.

4. Consider Necessitation. We have $K_{t_i}(\Omega) = \{\omega \in \Omega \mid t_i(\omega)(\Omega) = 1\} = \Omega$.

5. Consider Non-empty λ-Conjunction. Let $\mathcal{E}$ be such that $\mathcal{E} \subseteq \mathcal{D}$ and $0 < |\mathcal{E}| < \kappa$. Suppose that $\omega \in \bigcap_{E \in \mathcal{E}} K_i(E)$. Then, $t_i(\omega)(E) = 1$ for all $E \in \mathcal{E}$, and thus $1 = \min_{E \in \mathcal{E}} t_i(\omega)(E) \leq t_i(\omega)(\bigcap \mathcal{E})$. Thus, $\omega \in K_{t_i}(\bigcap \mathcal{E})$

6. Consider Truth Axiom. If $\omega \in K_{t_i}(E)$ then $t_i(\omega)(E) = 1$ and thus $\omega \in E$.

7. Consider Positive Introspection. If $\omega \in K_{t_i}(E)$ then $t_i(\omega)(E) = 1$. We then have $t_i(\omega)(\{\omega' \in \Omega \mid t_i(\omega')(E) = 1\}) = t_i(\omega)(K_{t_i}(E)) = 1$, leading to $\omega \in K_{t_i} K_{t_i}(E)$.

8. Consider Negative Introspection. If $\omega \in (\neg K_{t_i})(E)$ then $t_i(\omega)(E) = 0$. We then have $t_i(\omega)(\{\omega' \in \Omega \mid t_i(\omega')(E) = 0\}) = t_i(\omega)((\neg K_{t_i})(E)) = 1$, leading to $\omega \in K_{t_i}(\neg K_{t_i})(E)$.

9. The Kripke property follows because $b_{K_{t_i}} = b_{t_i}$.

Remark A.8 (Category Theoretical Equivalence of Definitions of Knowledge Spaces by Knowledge Operators and Knowledge-Type Mappings)**.** We denote by $\mathbf{K}_{oprt}$ and $\mathbf{K}_{type}$ the category of knowledge spaces given in terms of knowledge operators and knowledge-types, respectively. We define the following functor $T^{type}_{oprt} : \mathbf{K}_{oprt} \to \mathbf{K}_{type}$.

For any knowledge space $\overrightarrow{\Omega} = \langle (\Omega, \mathcal{D}), (K_i)_{i \in I}, \Theta \rangle$, we let $T^{type}_{oprt}(\overrightarrow{\Omega}) = \langle (\Omega, \mathcal{D}), (t_{K_i})_{i \in I}, \Theta \rangle$. For a knowledge morphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$, we let $T^{type}_{oprt}(\varphi) = \varphi$ as a (κ-measurable) mapping $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$.

Likewise, for any knowledge space $\overrightarrow{\Omega} = \langle (\Omega, \mathcal{D}), (t_i)_{i \in I}, \Theta \rangle$, we let $T^{oprt}_{type}(\overrightarrow{\Omega}) = \langle (\Omega, \mathcal{D}), (K_{t_i})_{i \in I}, \Theta \rangle$. For a knowledge morphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$, we let $T^{oprt}_{type}(\varphi) = \varphi$.

Clearly, we have $T^{oprt}_{type} \circ T^{type}_{oprt} = I_{oprt}$ and $T^{type}_{oprt} \circ T^{oprt}_{type} = I_{type}$, where I_{oprt} and I_{type} are the constant functors. In this way, we can establish the (category-theoretical) equivalence between the categories of knowledge spaces given in terms of knowledge operators and knowledge-types. We can establish the equivalence for the other combinations of primitives in a similar way.

A.4.2 Section 5.2

Proof of Proposition 11. 1. By definition, i knows t_i with respect to $\{\mathcal{M}(E) \mid E \in \mathcal{D}\}$ iff $\omega \in t_i^{-1}(\mathcal{M}(E))$ implies $t_i(\omega)(t_i^{-1}(\mathcal{M}(E))) = 1$ for each $E \in \mathcal{D}$.

2. Again, by definition, i knows t_i with respect to $\{\neg\mathcal{M}(E) \mid E \in \mathcal{D}\}$ iff $\omega \in \neg t_i^{-1}(\mathcal{M}(E))$ implies $t_i(\omega)(\neg t_i^{-1}(\mathcal{M}(E))) = 1$ for each $E \in \mathcal{D}$.

□

Proof of Proposition 12. 1. It follows from Truth Axiom that $\omega' \in b_{t_i}(\omega')$, because it implies that $\omega' \in E$ for all $E \in \mathcal{D}$ with $t_i(\omega')(E) = 1$. Hence, we have $\omega' \in b_{t_i}(\omega') \subseteq b_{t_i}(\omega)$ for all $\omega' \in (\uparrow t_i(\omega))$. The converse is immediate because $\omega \in [t_i(\omega)] \subseteq (\uparrow t_i(\omega)) \subseteq b_{t_i}(\omega)$. This means that for any $E \in \mathcal{D}$ with $t_i(\omega)(E) = 1$, $\omega \in b_{t_i}(\omega) \subseteq E$.

2. Suppose that $\omega' \in b_{t_i}(\omega)$. Now, suppose to the contrary that $\omega' \notin (\uparrow t_i(\omega))$, i.e., there is $F \in \mathcal{D}$ such that $t_i(\omega')(F) = 0 < 1 = t_i(\omega)(F)$. Then, by Positive Introspection, we have $t_i(\omega)(t_i^{-1}(\mathcal{M}_F)) = 1$, and thus $\omega' \in t_i^{-1}(\mathcal{M}_F)$, i.e., $t_i(\omega')(F) = 1$, a contradiction.

   Next, suppose that t_i satisfies the Kripke property. Assume that $b_{t_i}(\omega) \subseteq (\uparrow t_i(\omega))$. Suppose that $t_i(\omega)(E) = 1$. Then, $b_{t_i}(\omega) \subseteq E$. Now, in order to establish Positive Introspection, it is enough to show that $b_i(\omega') \subseteq E$ for all $\omega' \in b_{t_i}(\omega)$. Take any $\omega' \in b_{t_i}(\omega)$. Since $\omega' \in (\uparrow t_i(\omega))$ and $t_i(\omega)(b_{t_i}(\omega)) = 1$ holds, we have $t_i(\omega')(b_{t_i}(\omega)) = 1$. Thus, $b_{t_i}(\omega') \subseteq b_{t_i}(\omega) \subseteq E$.

3. The proof is analogous to the last assertion. Suppose that $\omega' \in b_{t_i}(\omega)$. Now, suppose to the contrary that $\omega' \notin (\downarrow t_i(\omega))$, i.e., there is $F \in \mathcal{D}$ such that $t_i(\omega)(F) = 0 < 1 = t_i(\omega')(F)$. Then, by Negative Introspection, we have $t_i(\omega)(\neg t_i^{-1}(\mathcal{M}_F)) = 1$, and thus $\omega' \in \neg t_i^{-1}(\mathcal{M}_F)$, i.e., $t_i(\omega')(F) = 0$, a contradiction.

   Next, suppose that t_i satisfies the Kripke property. Assume that $b_{t_i}(\omega) \subseteq (\downarrow t_i(\omega))$. Suppose that $t_i(\omega)(E) = 0$. Then, $b_{t_i}(\omega) \cap E^c \neq \emptyset$. Now, in order to establish Negative Introspection, it is enough to show that $b_i(\omega') \cap E^c \neq \emptyset$ for all $\omega' \in b_{t_i}(\omega)$. Take any $\omega' \in b_{t_i}(\omega)$. Since $\omega' \in (\downarrow t_i(\omega))$ and $t_i(\omega)(E) = 0$ holds, we have $t_i(\omega')(E) = 0$, i.e., $b_i(\omega') \cap E^c \neq \emptyset$.

4. By the previous assertions, we have $[t_i(\omega)] = (\uparrow t_i(\omega)) = b_{t_i}(\omega) \subseteq (\downarrow t_i(\omega))$. Thus, we show $(\downarrow t_i(\omega)) \subseteq [t_i(\omega)]$. Suppose not, i.e., there is $\omega' \in \Omega$ with $\omega' \in (\downarrow t_i(\omega))$ but $\omega' \notin (\uparrow t_i(\omega))$. Thus, $t_i(\omega')(F) = 0 < 1 = t_i(\omega)(F)$ for some $F \in \mathcal{D}$. Then, Negative Introspection implies that $t_i(\omega')((\neg t_i^{-1}(\mathcal{M}_F)) = 1$, so that $t_i(\omega)((\neg t_i^{-1}(\mathcal{M}_F)) = 1$. Truth Axiom then implies $\omega \in \neg t_i^{-1}(\mathcal{M}_F)$, i.e., $t_i(\omega)(F) = 0$, which is a contradiction to $t_i(\omega)(F) = 1$. Finally, when t_i satisfies the Kripke property, it is immediate that the converse is true.

□

A.4.3 Section 5.3

Proof of Proposition 13. First, since each player's knowledge satisfies Positive Introspection, it follows from Proposition 11 that each player i knows her own type-mapping $t_{K_i^*} : \Omega \to M(\Omega^*, \mathcal{D}^*)$ with respect to $\{\mathcal{M}([e]) \mid [e] \in \mathcal{D}^*\}$. Second, it follows from the assumptions of the statement that $C^* = K_i^*$ for all $i \in I$. □

The fact that players' knowledge operators are identical plays an important role in the proof of Proposition 13. The following proposition clarifies this point.

Proposition A.5 (Knowledge of Opponent's Type Implies Knowledgeability)**.** *Let $\overrightarrow{\Omega} := \langle (\Omega, \mathcal{D}), (t_i)_{i \in I}, \Theta \rangle$ be a knowledge space. Assume the same assumptions on knowledge of players i and j. Moreover, we impose Truth Axiom, Monotonicity, and*

Positive Introspection on them. Then, i knows j's knowledge-type mapping t_j with respect to $\{\mathcal{M}(E) \mid E \in \mathcal{D}\}$ iff $K_{t_j}(E) \subseteq K_{t_i}(E)$ for all $E \in \mathcal{D}$ (i.e., i is at least as knowledgeable as j).

Proof of Proposition A.5. If i knows j's knowledge-type mapping t_j, then we have $K_{t_j}(E) \subseteq K_{t_i}K_{t_j}(E) \subseteq K_{t_i}(E)$ for all $E \in \mathcal{D}$, where the second set inclusion follows from Truth Axiom of j and Monotonicity of i. Conversely, suppose that $K_{t_j}(E) \subseteq K_{t_i}(E)$ for all $E \in \mathcal{D}$. By Positive Introspection of j, we obtain $K_{t_j}(E) \subseteq K_{t_j}K_{t_j}(E) \subseteq K_{t_i}K_{t_j}(E)$ for all $E \in \mathcal{D}$. □

Proposition A.5 implies the following. Under the above assumptions, players' type mappings may not be necessarily commonly known in a given knowledge space $\overrightarrow{\Omega}$ when players' knowledge is different.

A.5 Section 6

A.5.1 Section 6.1

Proof of Remark 4. First, it follows from the condition of the statement that

$$\begin{aligned} K_{t_i}(\varphi^{-1}(E')) &= \{\omega \in \Omega \mid t_i(\omega)(\varphi^{-1}(E')) = 1\} = \{\omega \in \Omega \mid t'_i(\varphi(\omega))(E') = 1\} \\ &= \{\omega \in \Omega \mid \varphi(\omega) \in K'_{t'_i}(E')\} = \varphi^{-1}(K'_{t'_i}(E')). \end{aligned}$$

Conversely, it follows frm Condition (3) in Definition 3 that

$$\begin{aligned} t_{K_i}(\omega)(\varphi^{-1}(E')) = 1 &\text{ iff } \omega \in K_i(\varphi^{-1}(E')) = \varphi^{-1}(K'_i(E')) \\ &\text{ iff } \varphi(\omega) \in K'_i(E') \text{ iff } t'_{K'_i}(\varphi(\omega))(E') = 1. \end{aligned}$$

□

Remark A.9 ($(H, \mathcal{H})$ is a κ-Complete Algebra)**.** We show that $(H, \mathcal{H})$ is a κ-complete algebra. First, $H \in \mathcal{H}$ is obvious. Second, $\mathcal{H}$ is clearly closed under complementation. Indeed, if $(\pi^\alpha)^{-1}(E^\alpha) \in \mathcal{H}$ (with $\alpha < \kappa$) then $\neg(\pi^\alpha)^{-1}(E^\alpha) = (\pi^\alpha)^{-1}(\neg E^\alpha) \in \mathcal{H}$. Third, since κ is regular, it is also closed under κ-union/intersection. Take any subset A of ordinals $\{\alpha \mid \alpha < \kappa\}$ whose cardinal $|A|$ is less than κ. Then, its supremum γ has cardinal less than κ, given that κ is regular. Now, each $(\pi^\alpha)^{-1}(E^\alpha)$ is projected on H^γ. Then, since H^γ is closed under κ-intersection, the proof is complete.

Proof of Lemma 6. Part 1. We show by induction that $h^{-1}((\pi^\alpha|_{\Omega^*})^{-1}(E^\alpha)) = (h^\alpha)^{-1}(E^\alpha) \in \mathcal{D}$ for all $\alpha < \kappa$ and $E^\alpha \in \mathcal{H}^\alpha$. Let $\alpha = 0$. We have $(h^0)^{-1}(E^0) = \Theta^{-1}(E^0) \in \mathcal{D}$ for any $E^0 \in \mathcal{A}_S$. For a successor ordinal $\alpha = \beta + 1$, it is sufficient to show that $(t_i \circ (h^\beta)^{-1})^{-1}(\mathcal{M}^{(H^\beta, \mathcal{H}^\beta)}(E^\beta)) \in \mathcal{D}$ for each $E^\beta \in \mathcal{H}^\beta$ and $i \in I$. Fix $E^\beta \in \mathcal{H}^\beta$ and $i \in I$. Then, we have

$$\begin{aligned} (t_i \circ (h^\beta)^{-1})^{-1}(\mathcal{M}^{(H^\beta, \mathcal{H}^\beta)}(E^\beta)) &= \{\omega \in \Omega \mid t_i(\omega)((h^\beta)^{-1}(E^\beta)) = 1\} \\ &= t_i^{-1}(\mathcal{M}^{(\Omega, \mathcal{D})}((h^\beta)^{-1}(E^\beta))) \in \mathcal{D}. \end{aligned}$$

For a limit ordinal α, if $h^{-1}((\pi^\beta|_{\Omega^*})^{-1}(E^\beta)) = (h^\beta)^{-1}(E^\beta) \in \mathcal{D}$ for all $\beta < \alpha$, then it is clear that $h^{-1}((\pi^\alpha|_{\Omega^*})^{-1}(E^\alpha)) = (h^\alpha)^{-1}(E^\alpha) \in \mathcal{D}$.

Part 2. For notational ease, we denote $h = h_{\overrightarrow{\Omega}}$ and $h' = h_{\overrightarrow{\Omega'}}$. We show by induction that, for each ordinal $\alpha < \kappa$, $h^\alpha(\omega) = (h')^\alpha(\varphi(\omega))$ for each $\omega \in \Omega$. Let $\alpha = 0$. Since φ is a knowledge morphism, we have $h^0(\omega) = \Theta(\omega) = \Theta'(\varphi(\omega)) = (h')^0(\varphi(\omega))$ for each $\omega \in \Omega$. Let $\alpha = \beta + 1$ be a successor ordinal. Since we have

$$\begin{aligned} h^{\beta+1}(\omega) &= \left(h^\beta(\omega), t(\omega) \circ (h^\beta)^{-1}\right) \text{ and} \\ (h')^{\beta+1}(\varphi(\omega)) &= \left(h'^\beta(\varphi(\omega)), t'(\varphi(\omega)) \circ (h'^k)^{-1}\right), \end{aligned}$$

it suffices to show that $t(\omega) \circ (h^\beta)^{-1} = t'(\varphi(\omega)) \circ (h'^\beta)^{-1}$. Now, since φ is a knowledge morphism, we have, for each $i \in I$,

$$\begin{aligned} t_i'(\varphi(\omega)) \left((h'^\beta)^{-1}(\cdot)\right) &= t_i(\omega) \left(\varphi^{-1}\left((h'^\beta)^{-1}(\cdot)\right)\right) \\ &= t_i(\omega) \left(\left(h'^\beta \circ \varphi\right)^{-1}(\cdot)\right) = t_i(\omega) \left((h^\beta)^{-1}(\cdot)\right). \end{aligned}$$

Let α be a limit ordinal. Fix $\omega \in \Omega$. By the definitions of h^α and $(h')^\alpha$, it is immediate that $h^\alpha(\omega) = (h')^\alpha(\varphi(\omega))$ if $h^\beta(\omega) = (h')^\beta(\varphi(\omega))$ for all $\beta < \alpha$. The induction is complete. $\square$

Proof of Lemma 7. Fix $i \in I$. We prove the results in the following five steps. In the first step, we start with showing that t_i^* is a well-defined mapping on Ω^*. Fix $\omega^* \in \Omega^*$. We show that, for any ordinals (α, β) with $0 \leq \beta \leq \alpha < \kappa$, if $(\pi^\alpha|_{\Omega^*})^{-1}(E^\alpha) = (\pi^\beta|_{\Omega^*})^{-1}(F^\beta)$ for some $E^\alpha \in \mathcal{H}^\alpha$ and $E^\beta \in \mathcal{H}^\beta$, then $(\omega^*)_i^{\alpha+1}(E^\alpha) = (\omega^*)_i^{\beta+1}(F^\beta)$.

Observe first that, for any a knowledge space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $\omega^* = h(\omega)$, we have

$$t_i^*(\omega^*) \circ (\pi^\alpha|_{\Omega^*})^{-1} = (h(\omega))_i^{\alpha+1} = t_i(\omega) \circ (h^\alpha)^{-1} = t_i(\omega) \circ (\pi^\alpha|_{\Omega^*} \circ h)^{-1}. \tag{A.2}$$

Thus, we have

$$(\omega^*)_i^{\alpha+1}(E^\alpha) = t_i(\omega)((\pi^\alpha|_{\Omega^*})^{-1}(E^\alpha)) = t_i(\omega)((\pi^\beta|_{\Omega^*})^{-1}(F^\beta)) = (\omega^*)_i^{\beta+1}(F^\beta).$$

This equation holds irrespective of a choice of knowledge spaces.

In the second step, we establish Equation (12). Indeed, it follows from Equation (A.2) that, for each $\alpha < \kappa$ and $E^\alpha \in \mathcal{H}^\alpha$,

$$t_i^*(h(\omega))((\pi^\alpha|_{\Omega^*})^{-1}(E^\alpha)) = t_i(\omega)(h^{-1}((\pi^\alpha|_{\Omega^*})^{-1}(E^\alpha))).$$

In the third step, we show that t_i^* inherits all the logical properties of knowledge so that it is a mapping from Ω^* into $M(\Omega^*, \mathcal{D}^*)$. Indeed, the statement clearly holds because the inverse image h^{-1} commutes with set-algebraic operations.

1. Suppose that i's knowledge satisfies No-Contradiction Axiom. For any $\omega^* \in \Omega^*$, there are a knowledge space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $\omega^* = h(\omega)$, and thus $t_i^*(\omega^*)(\emptyset) = t_i(\omega)(h^{-1}(\emptyset)) = 0$.

2. Suppose that i's knowledge satisfies Consistency. Suppose that there are $\omega^* \in \Omega^*$ and $E^* \in \mathcal{D}^*$ such that $t_i^*(\omega^*)(E^*) = 1$. Then, there are a knowledge space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $\omega^* = h(\omega)$ and $t_i(\omega)(h^{-1}(E^*)) = 1$. By Consistency of t_i, we have $0 = t_i(\omega)(h^{-1}(\neg E^*)) = t_i(\omega)(\neg h^{-1}(E^*))$.

3. Suppose that i's knowledge satisfies Necessitation. For any $\omega^* \in \Omega^*$, there are a knowledge space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $\omega^* = h(\omega)$ and $1 = t_i(\omega)(\Omega) = t_i(\omega)(h^{-1}(\Omega^*)) = t_i^*(\omega^*)(\Omega^*)$.

4. Suppose that i's knowledge satisfies Monotonicity. Take any $\omega^* \in \Omega^*$ and $E^*, F^* \in \mathcal{D}^*$ with $E^* \subseteq F^*$. Now, there are a knowledge space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $\omega^* = h(\omega)$ and
$$t_i^*(\omega^*)(E^*) = t_i(\omega)(h^{-1}(E^*)) \leq t_i(\omega)(h^{-1}(F^*)) = t_i^*(\omega^*)(F^*),$$
where the weak inequality follows from $h^{-1}(E^*) \subseteq h^{-1}(F^*)$ and Monotonicity of t_i.

5. Suppose that i's knowledge satisfies Non-empty λ-Conjunction. Take any non-empty subset $\mathcal{F} \subseteq \mathcal{D}^*$ with $|\mathcal{F}| < \lambda$. Suppose that $t_i^*(\omega^*)(F^*) = 1$ for all $F^* \in \mathcal{F}$. Now, there are a knowledge space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $\omega^* = h(\omega)$ and $t_i(\omega)(h^{-1}(F^*)) = 1$ for all $F^* \in \mathcal{F}$. Since t_i satisfies Non-empty λ-Conjunction, we have $1 = t_i(\omega)(\bigcap_{F^* \in \mathcal{F}} h^{-1}(F^*)) = t_i(\omega)(h^{-1}(\bigcap \mathcal{F}))$, establishing $t_i^*(\omega^*)(\bigcap \mathcal{F}) = 1$.

In the fourth step, we show that $t_i^* : (\Omega^*, \mathcal{D}^*) \to (M(\Omega^*, \mathcal{D}^*), \mathcal{M}(\Omega^*, \mathcal{D}^*))$ is κ-measurable. Let $E^* = (\pi^\alpha|_{\Omega^*})^{-1}(E^\alpha)$ with $E^\alpha \in \mathcal{H}^\alpha$ and $\alpha < \kappa$. Then, we have
$$(t_i^*)^{-1}(\mathcal{M}(E^*)) = \{\omega^* \in \Omega^* \mid (\omega^*)_i^{\alpha+1}(E^\alpha) = 1\} = \{\omega^* \in \Omega^* \mid (\omega^*)_i^{\alpha+1} \in \mathcal{M}(E^\alpha)\}.$$

In the fifth step, we show that t_i^* inherits the introspective properties of knowledge.

1. Consider Truth Axiom. Fix $E^* \in \mathcal{D}^*$. If $t_i^*(\omega^*)(E^*) = 1$, then there exist a knowledge space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $\omega^* = h(\omega)$ and $1 = t_i^*(\omega^*)(E^*) = t_i(\omega)(h^{-1}(E^*))$. Now, Truth Axiom of t_i implies $\omega \in h^{-1}(E^*)$, and thus $\omega^* = h(\omega) \in E^*$.

2. Next, consider Positive Introspection. Fix $E^* \in \mathcal{D}^*$ and $\omega^* \in \Omega^*$ such that $t_i^*(\omega^*)(E^*) = 1$. Then, there are a knowledge space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $\omega^* = h(\omega)$, $1 = t_i^*(\omega^*)(E^*) = t_i(\omega)(h^{-1}(E^*))$, and $t_i^*(\omega^*)((t_i^*)^{-1}(\mathcal{M}_{E^*})) =$

$t_i(\omega)(h^{-1}((t_i^*)^{-1}(\mathcal{M}_{E^*})))$. Now, Positive Introspection of t_i implies that $1 = t_i(\omega)(t_i^{-1}(\mathcal{M}_{h^{-1}(E^*)}))$. Next, we show that $t_i^{-1}(\mathcal{M}_{h^{-1}(E^*)}) = h^{-1}((t_i^*)^{-1}(\mathcal{M}_{E^*}))$:

$$\begin{aligned}\omega \in t_i^{-1}(\mathcal{M}_{h^{-1}(E^*)}) &\text{ iff } t_i(\omega)(h^{-1}(E^*)) = 1 \text{ iff } t_i^*(\omega^*)(E^*) = 1 \\ &\text{ iff } h(\omega) = \omega^* \in (t_i^*)^{-1}(\mathcal{M}_{E^*}) \text{ iff } \omega \in h^{-1}((t_i^*)^{-1}(\mathcal{M}_{E^*})).\end{aligned}$$

Then, we obtain

$$1 = t_i(\omega)(t_i^{-1}(\mathcal{M}_{h^{-1}(E^*)})) = t_i(\omega)(h^{-1}((t_i^*)^{-1}(\mathcal{M}_{E^*}))) = t_i^*(\omega^*)((t_i^*)^{-1}(\mathcal{M}_{E^*})).$$

3. We show that t_i^* inherits Negative Introspection. Fix $E^* \in \mathcal{D}^*$ and $\omega^* \in \Omega^*$ with $t_i^*(\omega^*)(E^*) = 0$. Then, there are a knowledge space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $\omega^* = h(\omega)$ and $0 = t_i^*(\omega^*)(E^*) = t_i(\omega)(h^{-1}(E^*))$. Now, Negative Introspection of t_i implies that $1 = t_i(\omega)(\neg t_i^{-1}(\mathcal{M}_{h^{-1}(E^*)}))$. Then, it follows from the previous argument that $t_i^{-1}(\mathcal{M}_{h^{-1}(E^*)}) = h^{-1}((t_i^*)^{-1}(\mathcal{M}_{E^*}))$, and hence we obtain

$$1 = t_i(\omega)(\neg t_i^{-1}(\mathcal{M}_{h^{-1}(E^*)})) = t_i(\omega)(h^{-1}(\neg(t_i^*)^{-1}(\mathcal{M}_{E^*}))) = t_i^*(\omega^*)(\neg(t_i^*)^{-1}(\mathcal{M}_{E^*})).$$

4. We show that t_i^* inherits the Kripke property. It follows from Lemma A.2 that $h(b_{t_i}(\omega)) \subseteq b^*_{t_i^*}(h(\omega))$ for each $\omega \in \Omega$. Now, if $b^*_{t_i^*}(\omega^*) \subseteq E^*$, then there are a knowledge space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $\omega^* = h(\omega)$, and thus

$$b_{t_i}(\omega) \subseteq h^{-1}(h(b_{t_i}(\omega))) \subseteq h^{-1}(b^*_{t_i^*}(\omega^*)) \subseteq h^{-1}(E^*).$$

By the Kripke property of $\overrightarrow{\Omega}$, it follows that $1 = t_i(\omega)(h^{-1}(E^*)) = t_i^*(\omega^*)(E^*)$.

□

Proof of Lemma 8. We show by induction that $(h^*)^\alpha : \Omega^* \to H^\alpha$ is the projection $\pi^\alpha|_{\Omega^*}$ for all ordinal $\alpha < \kappa$, where note that $(h^*)^\alpha = \pi^\alpha \circ h^*$. For $\alpha = 0$, we have $(h^*)^0 = \pi^0|_{\Omega^*}$. Let $\alpha = \beta + 1$ be a successor ordinal. Then, for each ω^*, we have

$$\begin{aligned}(h^*)^{\beta+1}(\omega^*) &= ((h^*)^\beta(\omega^*), t^*(\omega^*) \circ ((h^*)^\beta)^{-1}) = (\pi^\beta|_{\Omega^*}(\omega^*), t^*(\omega^*) \circ (\pi^\beta|_{\Omega^*})^{-1}) \\ &= (\pi^\beta|_{\Omega^*}(\omega^*), (\omega^*)^{\beta+1}) = \pi^{\beta+1}|_{\Omega^*}(\omega^*),\end{aligned}$$

where $t^*(\omega^*) \circ (\pi^\beta|_{\Omega^*})^{-1} = (\omega^*)^{\beta+1}$ follows from Equation (A.2). Then, we get $(h^*)^{\beta+1} = \pi^{\beta+1}|_{\Omega^*}$. For a limit ordinal α, the statement clearly holds by construction. The induction is complete. □

Proof of Theorem 3. First, we establish that $\overrightarrow{\Omega^*}$ is universal. To that end, as is discussed in the main text, it is sufficient to show the uniqueness of a knowledge morphism $h : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$. The uniqueness, however, follows from Lemma 8 as in the proof of Theorem 1.

Second, Lemma 8 also implies that that $(\Theta^*, (t_i^*)_{i\in I}) : \Omega^* \to \Omega^{**}$ is injective. Thus, we show that $(\Theta^*, (t_i^*)_{i\in I}) : \Omega^* \to \Omega^{**}$ is surjective. For any $(s, (\mu_i)_{i\in I}) \in \Omega^{**}$, there are a knowledge space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that

$$(s, (\mu_i)_{i\in I}) = (\Theta(\omega), (t_i(\omega) \circ h^{-1})_{i\in I}) = (\Theta^*(h(\omega)), (t_i^*(h(\omega)))_{i\in I}).$$

□

A.5.2 Proof of Remark 5 (Section 6.1)

We prove the statement in Remark 5 by applying Lambek's lemma (Lambek [47]) in category theory. To that end, we represent a knowledge space in terms of (i) an underlying structure $(\Omega, \mathcal{D})$ and (ii) a tuple of mappings $(\Theta, (t_i)_{i\in I}) : \Omega \to S \times M(\Omega)^I$ in a category theoretical manner.

We first define underlying structures of knowledge spaces. We let $\mathbf{B}_\kappa$ be the category of κ-complete (Boolean) algebra (of sets). That is, each $\mathbf{B}_\kappa$-object is a κ-complete algebra $(\Omega, \mathcal{D})$, and each $\mathbf{B}_\kappa$-morphism is a κ-measurable mapping $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$.

Next, we define $(\Theta, (t_i)_{i\in I}) : (\Omega, \mathcal{D}) \to (F(\Omega, \mathcal{D}), \mathcal{F}(\Omega, \mathcal{D}))$ as a κ-measurable mapping as follows. Let $(\Omega, \mathcal{D})$ be a $\mathbf{B}_\kappa$-object. We define $F(\Omega, \mathcal{D}) := S \times M(\Omega, \mathcal{D})^I$ and $\mathcal{F}(\Omega, \mathcal{D}) := \mathcal{A}_S \times \mathcal{M}(\Omega, \mathcal{D})^I$. We often denote $(F(\Omega), \mathcal{F}(\mathcal{D})) = (F(\Omega, \mathcal{D}), \mathcal{F}(\Omega, \mathcal{D}))$.

Note that we can rewrite $\mathcal{F}(\Omega, \mathcal{D})$ as follows. To that end, we define the projections $\pi_S : F(\Omega, \mathcal{D}) \to S$ and $\pi_i : F(\Omega, \mathcal{D}) \to M(\Omega, \mathcal{D})$ for each $i \in I$. Then, we have

$$\mathcal{F}(\Omega, \mathcal{D}) = \mathcal{A}_\kappa \left(\{\pi_S^{-1}(E_S) \mid E_S \in \mathcal{A}_S\} \cup \bigcup_{i\in I} \{\pi_i^{-1}(\mathcal{M}_E) \mid E \in \mathcal{D}\} \right).$$

For each $\mathbf{B}_\kappa$-morphism $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$, we define a mapping $F(\varphi)$ defined on $F(\Omega, \mathcal{D})$ by

$$F(\varphi)(s, (\mu_i)_{i\in I}) := (s, (\mu_i \circ \varphi^{-1})_{i\in I}),$$

where note that $(\mu_i \circ \varphi^{-1})(E') := \mu_i(\varphi^{-1}(E'))$ for each $E' \in \mathcal{D}'$.

We show that $F : \mathbf{B}_\kappa \to \mathbf{B}_\kappa$ defined above is an endofunctor in the following three steps. In the first step, we show that $F(\varphi)$ is a mapping from $F(\Omega, \mathcal{D})$ into $F(\Omega', \mathcal{D}')$ for a given $\mathbf{B}_\kappa$-morphism $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$. This is, however, clear because the inverse image preserves the logical properties.

In the second step, we show that $F(\varphi) : (F(\Omega), \mathcal{F}(\mathcal{D})) \to (F(\Omega'), \mathcal{F}(\mathcal{D}'))$ is a $\mathbf{B}_\kappa$-morphism (a κ-measurable mapping) for each κ-measurable mapping $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$. It is enough to show that the inverse $(F(\varphi))^{-1}$ maps generators of $\mathcal{F}(\mathcal{D}')$ into $\mathcal{F}(\mathcal{D})$. To that end, consider first $\mathcal{A}_S$. For each $(\pi'_S)^{-1}(E_S)$ with $E_S \in \mathcal{A}_S$, we have

$$(F(\varphi))^{-1}((\pi'_S)^{-1}(E_S)) = \pi_S^{-1}(E_S) \in \mathcal{F}(\mathcal{D}).$$

Next, for each $(\pi_i')^{-1}(\{\mu' \in M(\Omega') \mid \mu'(E') = 1\})$ with $i \in I$ and $E' \in \mathcal{D}'$, we have

$$(F(\varphi))^{-1}((\pi_i')^{-1}(\{\mu' \in M(\Omega') \mid \mu'(E') = 1\})) = (\pi_i)^{-1}(\{\mu \in M(\Omega) \mid \mu(\varphi^{-1}(E')) = 1\}) \in \mathcal{F}(\mathcal{D}).$$

This establishes that $F(\varphi)$ is κ-measurable.

In the third step, we start with showing that $F(\psi \circ \varphi) = F(\psi) \circ F(\varphi)$. Indeed, we have

$$\begin{aligned} F(\psi \circ \varphi)(s, (\mu_i)_{i \in I}) &= (s, (\mu_i \circ (\psi \circ \varphi)^{-1})_{i \in I}) = (s, (\mu_i(\varphi^{-1}(\psi^{-1}(\cdot))))_{i \in I}) \\ &= F(\psi) \circ F(\varphi). \end{aligned}$$

Next, it is easily seen that $F(\mathrm{id}_\Omega) = \mathrm{id}_{F(\Omega)}$. Thus, $F : \mathbf{B}_\kappa \to \mathbf{B}_\kappa$ is an endofunctor.

Recall that a knowledge space of I on $(S, \mathcal{A}_S)$ (in terms of knowledge-type mappings) is a tuple $\langle (\Omega, \mathcal{D}), (t_i)_{i \in I}, \Theta \rangle$ where $(\Omega, \mathcal{D})$ is a κ-measurable space (i.e., a $\mathbf{B}_\kappa$-object) and $(\Theta, (t_i)_{i \in I}) : (\Omega, \mathcal{D}) \to (F(\Omega, \mathcal{D}), \mathcal{F}(\Omega, \mathcal{D}))$ is a κ-measurable mapping (i.e., a $\mathbf{B}_\kappa$-morphism) such that each type mapping t_i satisfies the required introspective (i.e., Truth Axiom, Positive Introspection, and Negative Introspection) and Kripke properties.[A.2]

A knowledge morphism $\varphi : \langle (\Omega, \mathcal{D}), (t_i)_{i \in I}, \Theta \rangle \to \langle (\Omega', \mathcal{D}'), (t_i')_{i \in I}, \Theta' \rangle$ (where both knowledge spaces reside in the same class of knowledge spaces) can be seen as a κ-measurable mapping $\varphi : (\Omega, \mathcal{D}) \to (\Omega, \mathcal{D})$ such that $(\Theta', (t_i')_{i \in I}) \circ \varphi = F(\varphi) \circ (\Theta, (t_i)_{i \in I})$. In other words,

$$(\Theta'(\varphi(\omega)), (t_i'(\varphi(\omega))(\cdot))_{i \in I}) = (\Theta(\omega), (t_i(\omega)(\varphi^{-1}(\cdot)))_{i \in I}) \text{ for each } \omega \in \Omega.$$

In the language of category theory, a pair $\langle (\Omega, \mathcal{D}), (\Theta, (t_i)_{i \in I}) \rangle$ is a coalgebra over the endofunctor F. Thus, the category of knowledge(-type) spaces is seen as the full subcategory of F-coalgebras such that each t_i satisfies the given introspective and/or Kripke properties.

Now, we show that if no introspective property nor the Kripke property is imposed in a given class of knowledge spaces then the universal knowledge space $\overrightarrow{\Omega^*}$ studied in Section 6.1 is isomorphic to $S \times M(\Omega^*, \mathcal{D}^*)^I$.

Proof of Remark 5. The assertion simply follows from the Lambek lemma (Lambek [47]) in category theory. For completeness, we provide a proof below.

$$\begin{tikzcd}
(\Omega^*, \mathcal{D}^*) \arrow[r, "(\Theta^*, t^*)"] \arrow[d, "(\Theta^*, t^*)"'] & F(\Omega^*, \mathcal{D}^*) \arrow[r, "h"] \arrow[d, "F(\Theta^*, t^*)"'] & (\Omega^*, \mathcal{D}^*) \arrow[d, "(\Theta^*, t^*)"] \\
F(\Omega^*, \mathcal{D}^*) \arrow[r, "F(\Theta^*, t^*)"'] & F^2(\Omega^*, \mathcal{D}^*) \arrow[r, "F(h)"'] & F(\Omega^*, \mathcal{D}^*)
\end{tikzcd}$$

[A.2] We simply denote the product mapping by $(\Theta, t)(\omega) := (\Theta, (t_i)_{i \in I})(\omega) := (\Theta(\omega), (t_i(\omega))_{i \in I}) \in F(\Omega) = S \times M(\Omega)^I$ for each $\omega \in \Omega$.

First, $\langle (F(\Omega^*), \mathcal{F}(\mathcal{D}^*)), F(\Theta^*, (t_i^*)_{i \in I}) \rangle$ is a knowledge space. Since $\overrightarrow{\Omega^*}$ is universal, let $h : F(\Omega^*) \to \Omega^*$ be the description map. Since there is a unique knowledge morphism from $\overrightarrow{\Omega^*}$ into itself and since it is the identity map, we obtain $h \circ (\Theta^*, t^*) = \mathrm{id}_{\Omega^*}$. Next, we also have

$$F(h) \circ F(\Theta^*, t^*) = F(h \circ (\Theta^*, t^*)) = F(\mathrm{id}_{\Omega^*}) = \mathrm{id}_{F(\Omega^*, \mathcal{D}^*)}.$$

Then, we get $(\Theta^*, t^*) \circ h = F(h) \circ F(\Theta^*, t^*) = \mathrm{id}_{F(\Omega^*, \mathcal{D}^*)}$. Hence, $(\Theta^*, t^*) := (\Theta^*, (t_i^*)_{i \in I})$ is a knowledge isomorphism and its inverse is h. $\square$

A.5.3 Section 6.2 (Proof of Theorem 4)

Proof of Theorem 4. Step 1. First, consider the universal knowledge space $\overrightarrow{\Omega^*}$ established in Section 6.1. By definition, Ω^* is a subset of H. The κ-complete algebra $\mathcal{D}^*$ and the mapping Θ^* are defined as in Definition 17. Second, it is also clear that Ω^* respects the required introspective properties of knowledge. Third, we have already shown that Ω^* is coherent in establishing that $\overrightarrow{\Omega^*}$ is universal.

Step 2. Let Ω be a coherent subset of H which respects the required introspective properties. First, we show that $\langle (\Omega, \mathcal{D}), (t_i)_{i \in I}, \Theta \rangle$ is a knowledge space in a given category. To do so, it is enough to show that t_i satisfies the required logical properties of knowledge.

1. Consider No-Contradiction Axiom. We have $t_i(s, \mu)(\emptyset) = \mu_i^\alpha(\emptyset) = 0$.

2. Consider Necessitation. We have $t_i(s, \mu)(\Omega) = \mu_i^\alpha(H^\alpha) = 1$.

3. Consider Consistency. If $t_i(s, \mu)((\pi^\alpha|_\Omega)^{-1}(E^\alpha)) = 1$ and $t_i(s, \mu)(\neg(\pi^\alpha|_\Omega)^{-1}) = 1$, then $\mu_i^\alpha(E^\alpha) = 1$ and $\mu_i^\alpha(\neg E^\alpha) = 1$, which contradicts the assumption that μ_i^α is a knowledge-type satisfying Consistency (i.e., $\mu_i^\alpha \in M(H^\alpha, \mathcal{H}^\alpha)$).

4. Consider Monotonicity. Suppose that $(\pi^\alpha|_\Omega)^{-1}(E^\alpha) \subseteq (\pi^\beta|_\Omega)^{-1}(F^\beta)$. Without loss of generality, we can assume as if $\alpha = \beta$. Since $E^\alpha \subseteq F^\beta$, we have

$$t_i(s, \mu)((\pi^\alpha|_\Omega)^{-1}(E^\alpha)) = \mu_i^\alpha(E^\alpha) \leq \mu_i^\beta(F^\beta) = t_i(s, \mu)((\pi^\beta|_\Omega)^{-1}(F^\beta)).$$

5. Consider Non-empty λ-Conjunction. Without loss of generality, suppose that $t_i(t)((\pi^\alpha|_\Omega)^{-1}(E^\alpha)) = 1$ for all $E^\alpha \in \mathcal{E}$, where $\mathcal{E}$ is a non-empty subset of $\mathcal{H}^\alpha$ with $|\mathcal{E}| < \lambda$. Then, we obtain

$$t_i(s, \mu)(\bigcap_{E^\alpha \in \mathcal{E}} (\pi^\alpha|_\Omega)^{-1}(E^\alpha)) = t_i(s, \mu)((\pi^\alpha|_\Omega)^{-1}(\bigcap \mathcal{E})) = \mu_i^\alpha(\bigcap \mathcal{E}) = 1.$$

Second, we show that the description map $h : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ is an inclusion map. For $\alpha = 0$, we have $h^0 = \pi^0|_\Omega$. For a successor ordinal $\alpha = \beta + 1$, since $t(s,\mu) \circ (\pi^\beta|_\Omega)^{-1} = \mu^\beta$, we have $h^\alpha = \pi^\alpha|_\Omega$. For a limit ordinal α, by construction, if $h^\beta = \pi^\beta|_\Omega$ for all $\beta < \alpha$ then $h^\alpha = \pi^\alpha|_\Omega$.

□

A.6 Section 7

Proof of Theorem 5. Construct Ω^* by collecting all the dynamic knowledge-belief spaces as in the proof of Theorem 1, where note the following. First, Ω^* is not empty because we can consider a dynamic knowledge-belief space $\overrightarrow{\{s\}}$ as in Section 3.1. Second, it follows from Lemma A.1 that the space $\overrightarrow{\Omega^*}$ satisfies the specified properties of knowledge and belief including perfect recall (when it is assumed). □

Proof of Theorem 6. First, our entire arguments in Section 3.1 establish a universal knowledge-unawareness space, where the universal knowledge-unawareness space inherits the properties of knowledge and unawareness by Lemma A.1. It is non-empty because we can consider a knowledge-unawareness space $\overrightarrow{\{s\}}$ as in Section 3.1.

Second, suppose that, in a given category, there is a knowledge-unawareness space $\overrightarrow{\Omega}$ such that $U_i([\![e]\!]_{\overrightarrow{\Omega}}) \neq \emptyset$ for some $i \in I$ and $e \in \mathcal{L}$. Then, there is $\omega \in U_i([\![e]\!]_{\overrightarrow{\Omega}}) = D^{-1}(U_i^*([e]))$ and thus $D(\omega) \in U_i^*([e])$. Hence, $\overrightarrow{\Omega^*}$ is non-trivial. □

A.7 Further Knowledge Representations in Knowledge Spaces (Supplementary Appendix to Section 4)

A.7.1 Generalized Posibility Correspondences

The Kripke property certainly generalizes the idea of the previous possibility correspondence models of knowledge in the sense that each player's knowledge at each state ω is characterized by her information set $b_{K_i}(\omega)$. The information set, however, is not necessarily an event. Moreover, a player whose knowledge satisfies the Kripke property is logically omniscient.[A.3]

We generalize standard possibility correspondence models in the following ways. First, we retain a spirit of the Kripke property in the sense that each player's knowledge is logically entailed from her information. Second, we require information that represents each player's knowledge to be an object of knowledge. Third, while the first requirement pre-supposes Monotonicity, we aim to dispense with Arbitrary Conjunction (including Necessitation) endemic in the Kripke property.

[A.3] DLR [21] show that logical omniscience inherent in the standard possibility correspondence models precludes sensible unawareness under certain conditions.

Specifically, a (generalized) possibility correspondence associates, with each sate of the world ω, a collection of events $\mathcal{B}_i(\omega)$ that can be a source/generator of knowledge at the state ω in the following sense: an agent knows an event E at a state ω if there is an event $F \in \mathcal{B}_i(\omega)$ which is contained in E. In order to distinguish between standard and generalized possibility correspondences, hereafter, we call such a generalized possibility correspondence to be an *information correspondence*. While we require Monotonicity by the very definition, it turns out that this information correspondence approach works in any domain.

At the conceptual level, $\mathcal{B}_i(\omega)$ is no longer interpreted as the set of states considered possible at ω. Each collection $\mathcal{B}_i(\omega)$ of events can be understood as the collection of information available to player i at state ω. Thus, we say that E is i's information set at ω if $E \in \mathcal{B}_i(\omega)$.

Before we go on to the formal analysis, we discuss the following two closely related approaches. First, Fagin and Halpern [27] take the "society-of-mind" (or "local reasoning") approach, where an agent, endowed with a collection of possibility sets $\mathcal{B}_i(\omega)$ at each ω, considers one possibility set possible at each time when she makes inferences.[A.4]

Second, Doignon and Falmagne [22, 23] study what they call "surmise systems" in the psychology literature. A surmise system $\mathcal{B}_i$ encodes all possible (thus not necessarily unique) ways to making inferences about the world at each state.[A.5] See Section A.7.4 for more details.

We start with the following preliminary notation. Throughout the subsection, fix a κ-complete algebra $(\Omega, \mathcal{D})$. For any subset Γ of $\mathcal{D}$, we define

$$\uparrow \Gamma := \{E \in \mathcal{D} \mid \text{there is } F \in \Gamma \text{ with } F \subseteq E\}.$$

If Γ is player's information (at a particular state) then $E \in \uparrow \Gamma$ means that there is information $F \in \Gamma$ which entails E. Note that $\uparrow \Gamma$ is closed under Monotonicity in the sense that $\uparrow\uparrow \Gamma \subseteq \uparrow \Gamma$.

An information correspondence on $(\Omega, \mathcal{D})$ is a mapping $\mathcal{B}_i$ from Ω into a subset of $\mathcal{P}(\mathcal{D})$ which satisfies the assumptions on i's knowledge and the following regularity

[A.4] See also the references therein. Especially, both approaches are related to the neighborhood (Montague-Scott) systems approach in modal logic, provided that Monotonicity is assumed (Fagin and Halpern [27, Footnote 9]). In our framework, this approach amounts to specifying the collection of events that a player i knows at each state ω. Our exposition in this subsection works by slight modifications if each player's knowledge is given by the collection of events that she knows at each state.

[A.5] In light of inferences, our idea is also related to that of Shin [80]. He identifies the notion of knowledge with the logical provability, where a player knows an event when she can prove it from her "basic knowledge" through use of propositional logic. In our framework, we can also interpret the knowledge of an E as the fact that she can "prove" E from one of her possibility set $F \in \mathcal{B}_i(\omega)$ (which could be incorrect in the sense that $\omega \notin F$), where provability is simply taken as the set inclusion in the form of $F \subseteq E$.

condition: for each $E \in \mathcal{D}$,

$$K_{\mathcal{B}_i}(E) := \{\omega \in \Omega \mid E \in\uparrow \mathcal{B}_i(\omega)\} \in \mathcal{D}. \tag{A.3}$$

$K_{\mathcal{B}_i}(E)$ is interpreted as the set of states at which player i has information to support E. Thus, we define the knowledge operator $K_{\mathcal{B}_i} : \mathcal{D} \to \mathcal{D}$ derived from the information correspondence $\mathcal{B}_i$ through Equation (A.3).

We have three remarks regarding the definition of $K_{\mathcal{B}_i}$. First, observe that $\uparrow \mathcal{B}_i(\omega)$ is exactly the collection of events that player i knows at ω. Put differently, $\omega \in K_{\mathcal{B}_i}(E)$ iff $E \in\uparrow \mathcal{B}_i(\omega)$.

Second, the event that i considers E possible can be written as

$$L_{\mathcal{B}_i}(E) = \{\omega \in \Omega \mid F \cap E^c \neq \emptyset \text{ for all } F \in \mathcal{B}_i(\omega)\}.$$

That is, it is the set of states ω such that i's information is not inconsistent with E.

Third, if $\mathcal{B}_i$ is singleton-valued (i.e., if $\mathcal{B}_i(\cdot) = \{b_i(\cdot)\}$) then it reduces to the (standard) possibility correspondence such (i) that each information/possibility set $b_i(\omega)$ is an event and (ii) that $\mathcal{B}_i$ satisfies the regularity condition (i.e., $\{\omega \in \Omega \mid b_i(\omega) \subseteq E\} \in \mathcal{D}$ for each $E \in \mathcal{D}$). We can dispense with Non-empty Conjunction by having multiple possibility sets while we can dispense with Necessitation by allowing the correspondence to be empty-valued. Suppose, for example, that $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and $\mathcal{D} = \mathcal{P}(\Omega)$. Let $\mathcal{B}_i$ be such that $\mathcal{B}_i(\omega_1) = \{\{\omega_1, \omega_2\}, \{\omega_1, \omega_3\}\}$ and $\mathcal{B}_i(\omega_2) = \mathcal{B}_i(\omega_3) = \emptyset$. Then an agent i knows $\{\omega_1, \omega_2\}$ and $\{\omega_1, \omega_3\}$ at state ω_1 but shes does not (cannot) know $\{\omega_1, \omega_2\} \cap \{\omega_1, \omega_3\}$ at that state. At state ω_j (with $j \in \{2, 3\}$), she does not know anything at all, and thus $K_{\mathcal{B}_i}(\Omega) = \{\omega_1\}$.

As we have established the equivalence between knowledge-operator and knowledge-type approaches in Section 5.1, we formalize a class of knowledge spaces (that satisfy Monotonicity) in terms of information correspondences for a given notion of knowledge in the following three steps.

The first step is to define a collection of information sets that information correspondences can take (i.e., the codomain of information correspondences $\mathcal{B}_i$, which is a subset of $\mathcal{P}(\mathcal{D})$). The arguments are parallel to defining the set of knowledge-types $M(\Omega, \mathcal{D})$ in the knowledge-type mapping approach. To that end, we call a subset Γ of $\mathcal{P}(\mathcal{D})$ to be an information collection.

Definition A.1 (Logical Properties of an Information Collection)**.** *Let Γ be a subset of $\mathcal{P}(\mathcal{D})$.*

1. *Γ satisfies No-Contradiction Axiom if $\emptyset \not\in\uparrow \Gamma$.*
2. *Γ is serial (satisfies Consistency) if $E^c \not\in\uparrow \Gamma$ for any $E \in\uparrow \Gamma$.*
3. *Γ satisfies Necessitation if $\Omega \in\uparrow \Gamma$.*
4. *Γ satisfies Non-empty λ-Conjunction if $\uparrow \Gamma$ is closed under non-empty λ-intersection (i.e., $\bigcap \mathcal{F} \in\uparrow \Gamma$ for any $\mathcal{F} \subseteq\uparrow \Gamma$ with $0 < |\mathcal{F}| < \lambda(\leq \kappa)$).*

We formalize the logical properties in terms of $\uparrow \Gamma$ (instead of the primitive Γ), because $\uparrow \Gamma$ corresponds to the collection of events that an agent knows by making inferences from Γ. Thus, it is clear that the above definition embodies our intended definitions of logical properties of knowledge. For example, Consistency of Γ means that if an agent "Γ-knows" E then she does not "Γ-know" its negation E^c. Another reason is that the above definition can also be used when each player's knowledge is expressed directly in terms of the collection of events that she knows at each state. Yet we provide a straightforward restatement of logical properties in terms of Γ.

Proposition A.6 (Logical Properties of an Information Collection)**.** *Let Γ be a subset of $\mathcal{P}(\mathcal{D})$.*

1. *Γ satisfies No-Contradiction Axiom iff $\emptyset \notin \Gamma$.*
2. *Γ is satisfies Consistency iff $E \cap F \neq \emptyset$ for any $E, F \in \Gamma$.*
3. *Γ satisfies Necessitation iff $\Gamma \neq \emptyset$.*
4. *Γ satisfies Non-empty λ-Conjunction iff, for any $\mathcal{F} \subseteq \Gamma$ with $0 < |\mathcal{F}| < \lambda (\leq \kappa)$, there is $F \in \Gamma$ with $F \subseteq \bigcap \mathcal{F}$.*

Proposition A.6 states the following. If a given information collection Γ does not contain the empty set ("a contradiction") then no contradiction is entailed from Γ. Γ satisfies Consistency when any pair of information $(E, F) \in \Gamma^2$ is not contradictory. Γ satisfies Necessitation when it contains some information in the sense that any tautology is inferred from it. Γ satisfies Non-empty Conjunction when, for any given family of information, the information collection Γ is rich enough to have another information which implies the conjunction of the given family.

In the second step, given a pre-determined concept of knowledge, let $S(\Omega, \mathcal{D}) (= S(\Omega))$ be a family of all information collections which satisfy all the implied logical properties. Given such family $S(\Omega)$, we introduce a κ-complete algebra $\mathcal{S}(\Omega, \mathcal{D}) (= \mathcal{S}(\mathcal{D}))$ by $\mathcal{S}(\mathcal{D}) := \mathcal{A}_\kappa(\{\mathcal{S}_E \in \mathcal{P}(S(\Omega)) \mid E \in \mathcal{D}\})$, where

$$\mathcal{S}_E := \{\Gamma \in S(\Omega) \mid E \in \uparrow \Gamma\}.$$

As the third and last step, we define the properties of knowledge on a correspondence $\mathcal{B}_i : (\Omega, \mathcal{D}) \to (S(\Omega), \mathcal{S}(\mathcal{D}))$. First, for all the logical properties of knowledge except for Monotonicity, we say that $\mathcal{B}_i$ satisfies a given logical property if the information set $\mathcal{B}_i(\omega)$ satisfies it for all $\omega \in \Omega$. For example, $\mathcal{B}_i$ satisfies *No-Contradiction Axiom* if each $\mathcal{B}_i(\omega)$ satisfies it (i.e., $\emptyset \notin \uparrow \mathcal{B}_i(\omega)$). Consistency, Necessitation, and Non-empty λ-Conjunction of $\mathcal{B}_i$ are defined in the similar way. Next, we define the introspective and Kripke properties of $\mathcal{B}_i$ as follows.

Definition A.2 (Introspective Properties of an Information Correspondence)**.** *Fix a κ-measurable mapping $\mathcal{B}_i : (\Omega, \mathcal{D}) \to (S(\Omega), \mathcal{S}(\mathcal{D}))$.*

1. *$\mathcal{B}_i$ is reflexive (satisfies Truth Axiom) if, $E \in \uparrow \mathcal{B}_i(\omega)$ implies $\omega \in E$ (i.e., $\uparrow \mathcal{B}_i(\omega) \subseteq \{E \in \mathcal{D} \mid \omega \in E\}$) for any $\omega \in \Omega$.*

2. *$\mathcal{B}_i$ is transitive (satisfies Positive Introspection) if $\{\omega' \in \Omega \mid E \in \uparrow \mathcal{B}_i(\omega')\} \in \uparrow \mathcal{B}_i(\omega)$ for any $\omega \in \Omega$ and $E \in \uparrow \mathcal{B}_i(\omega)$.*

3. *$\mathcal{B}_i$ is Euclidean (satisfies Negative Introspection) if the following holds: if $E \notin \uparrow \mathcal{B}_i(\omega)$ for some $(\omega, E) \in \Omega \times \mathcal{D}$ then $\{\omega' \in \Omega \mid E \notin \uparrow \mathcal{B}_i(\omega')\} \in \uparrow \mathcal{B}_i(\omega)$.*

4. *$\mathcal{B}_i$ satisfies the Kripke property if $\uparrow \mathcal{B}_i(\omega) = \{E \in \mathcal{D} \mid \bigcap \uparrow \mathcal{B}_i(\omega) \subseteq E\}$ for each $\omega \in \Omega$.*

Again, given that the fact that $E \in \uparrow \mathcal{B}_i(\omega)$ is intended to mean that E is known at ω, it is clear that the above definition embodies the intended introspective properties. Note, however, that $\bigcap \uparrow \mathcal{B}_i(\omega)$ may not be an event. In other words, it is not necessarily the case that $\uparrow \mathcal{B}_i(\omega)$ consists of the super-sets of $\bigcap \uparrow \mathcal{B}_i(\omega)$. Now, we restate the introspective properties in terms of $\mathcal{B}_i$.

Proposition A.7 (Introspective Properties of an Information Correspondence)**.** *Fix a κ-measurable mapping $\mathcal{B}_i : (\Omega, \mathcal{D}) \to (S(\Omega), \mathcal{S}(\mathcal{D}))$.*

1. *$\mathcal{B}_i$ is reflexive iff $E \in \mathcal{B}_i(\omega)$ implies $\omega \in E$ for any $\omega \in \Omega$.*

2. *$\mathcal{B}_i$ is transitive iff, for any $(\omega, E) \in \Omega \times \mathcal{D}$ with $E \in \mathcal{B}_i(\omega)$, there is $F \in \mathcal{B}_i(\omega)$ such that if $\omega' \in F$ then there is $E' \in \mathcal{B}_i(\omega')$ with $E' \subseteq E$.*

3. *$\mathcal{B}_i$ is Euclidean iff the following holds. If there is $(\omega, E) \in \Omega \times \mathcal{D}$ such that $E^c \cap F \neq \emptyset$ for all $F \in \mathcal{B}_i(\omega)$, then there is $F' \in \mathcal{B}_i(\omega)$ such that if $\omega' \in F'$ then $E^c \cap F \neq \emptyset$ for any $F \in \mathcal{B}_i(\omega')$.*

We interpret Proposition A.7 as follows. First, $\mathcal{B}_i$ is reflexive when i's information is always correct at each state. Second, $\mathcal{B}_i$ is transitive when, for any information E available to i at a state, there is another information F available to her at the same state (which can possibly be E itself) such that E is always supported as long as F is true. Third, $\mathcal{B}_i$ is Euclidean when, if i considers E possible at a state for some event E, then she has information F at the same state implying that i considers E possible whenever F is true.

As a further remark, assume the standard case where $\mathcal{B}_i(\cdot) = \{b_i(\cdot)\}$. First, $\mathcal{B}_i$ is reflexive iff $\omega \in b_i(\omega)$. Second, $\mathcal{B}_i$ is transitive iff $b_i(\omega') \subseteq b_i(\omega)$ for any $\omega' \in b_i(\omega)$. Third, $\mathcal{B}_i$ is Euclidean iff $\omega' \in b_i(\omega)$ implies $b_i(\omega) \subseteq b_i(\omega')$.[A.6]

In conclusion, given pre-determined assumptions on players' knowledge, a player i's information correspondence is a κ-measurable mapping $\mathcal{B}_i : (\Omega, \mathcal{D}) \to (S(\Omega), \mathcal{S}(\mathcal{D}))$ which satisfies the pre-determined properties of knowledge.

[A.6]Clearly, the Euclidean property of $\mathcal{B}_i$ is given as follows: if there is $E \in \mathcal{D}$ such that $E^c \cap b_i(\omega) \neq \emptyset$ and if $\omega' \in b_i(\omega)$, then $E^c \cap b_i(\omega') \neq \emptyset$. This is obviously equivalent to the statement in question.

Now, we prove the equivalence between knowledge-operator and information correspondence approaches under Monotonicity. First, if an information correspondence $\mathcal{B}_i : (\Omega, \mathcal{D}) \to (S(\Omega), \mathcal{S}(\mathcal{D}))$ is given, then it is clear that $K_{\mathcal{B}_i} : \mathcal{D} \to \mathcal{D}$ inherits the all the properties of knowledge imposed on $\mathcal{B}_i$.

Second, we define an information correspondence from a given knowledge operator $K_i : \mathcal{D} \to \mathcal{D}$ which (at least) satisfies Monotonicity in such a way that the knowledge operator induced by $\mathcal{B}_i$ coincides with the original knowledge operator. Formally, we say that, for a given knowledge operator $K_i : \mathcal{D} \to \mathcal{D}$, an information correspondence $\mathcal{B}_i : (\Omega, \mathcal{D}) \to (S(\Omega), \mathcal{S}(\mathcal{D}))$ *induces* K_i (or $\mathcal{B}_i$ is a *generator* of K_i) if $K_i = K_{\mathcal{B}_i}$.

The simplest way to find a generator of the given knowledge operator K_i (which satisfies Monotonicity) is to consider the information correspondence $\mathcal{B}_{K_i} : (\Omega, \mathcal{D}) \to (S(\Omega), \mathcal{S}(\mathcal{D}))$ defined by

$$\mathcal{B}_{K_i}(\omega) := \{E \in \mathcal{D} \mid \omega \in K_i(E)\} \text{ for each } \omega \in \Omega. \tag{A.4}$$

Since K_i satisfies Monotonicity, it follows that $K_i = K_{\mathcal{B}_{K_i}}$. We summarize the entire arguments in the following.

Proposition A.8 (Equivalence Between Information Correspondence and Knowledge Operator under Monotonicity). *Let $\overrightarrow{\Omega}$ be a κ-knowledge space such that Monotonicity is assumed for every player.*

1. *Let $\mathcal{B}_i : (\Omega, \mathcal{D}) \to (S(\Omega), \mathcal{S}(\mathcal{D}))$ be an information correspondence. Then, $K_{\mathcal{B}_i}$ inherits the properties of knowledge imposed on $\mathcal{B}_i$. Furthermore, $\uparrow \mathcal{B}_i(\omega) = \mathcal{B}_{K_{\mathcal{B}_i}}(\omega)$ for all $\omega \in \Omega$, where $\mathcal{B}_{K_{\mathcal{B}_i}}$ is defined as in Equation (A.4).*
2. *Given a knowledge operator K_i, let $\mathcal{B}_i : (\Omega, \mathcal{D}) \to (S(\Omega), \mathcal{S}(\mathcal{D}))$ be a generator of K_i. Then, $\mathcal{B}_i$ inherits the properties of knowledge imposed on K_i. Also, $\mathcal{B}_{K_i}$ defined as in Equation (A.4) is one generator of K_i.*

We make three additional remarks. First, if K_i satisfies Positive Introspection as well as Monotonicity, then we can restrict attention to self-evident events so that the following information correspondence $\mathcal{B}_{K_i}$ is also a generator:

$$\mathcal{B}_{K_i}(\omega) = \{E \in \mathcal{D} \mid \omega \in K_i(E) \text{ and } E \subseteq K_i(E)\}.$$

Second, a given knowledge operator K_i generally has multiple generators. Any information correspondences $\mathcal{B}_i$ and $\mathcal{B}'_i$ satisfying $\uparrow \mathcal{B}_i(\cdot) = \uparrow \mathcal{B}'_i(\cdot)$ induce the same knowledge operator $K_{\mathcal{B}_i} = K_{\mathcal{B}'_i}$.

Third, information correspondences characterize knowledgeability as follows. Let $\mathcal{B}_i$ and $\mathcal{B}_j$ be generators of K_i and K_j, respectively. Then, since $\{E \in \mathcal{D} \mid \omega \in K_i(E)\} = \{E \in \mathcal{D} \mid E \in \uparrow \mathcal{B}_i(\omega)\}$, we have $K_i(\cdot) \subseteq K_j(\cdot)$ iff $\uparrow \mathcal{B}_i(\cdot) \subseteq \uparrow \mathcal{B}_j(\cdot)$.

We conclude this subsection by characterizing knowledge morphisms in terms of information correspondences.

Proposition A.9 (Characterization of Knowledge Morphisms in terms of Information Correspondences). *Let $\varphi : \Omega \to \Omega'$ be a mapping between knowledge spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$ satisfying Monotonicity. Condition (3) in Definition 3 is written in terms of information correspondences as follows. Fix $\omega \in \Omega$.*

1. *For any $E' \in \mathcal{B}'_i(\varphi(\omega))$, there is $E \in \mathcal{B}_i(\omega)$ such that $\varphi(E) \subseteq E'$; and*

2. *For any $E \in \mathcal{B}_i(\omega)$ and $E' \in \mathcal{D}'$ with $\varphi(E) \subseteq E'$, there is $F' \in \mathcal{B}'_i(\varphi(\omega))$ such that $F' \subseteq E'$.*

As an immediate corollary, if $\mathcal{B}_i(\cdot) = \{b_i(\cdot)\}$, then the above conditions are rewritten as: (i) $\varphi(b_i(\omega)) \subseteq b'_i(\varphi(\omega))$; and (ii) $b'_i(\varphi(\omega)) \subseteq E'$ for any $E' \in \mathcal{D}'$ with $\varphi(b_i(\omega)) \subseteq E'$.

A.7.2 Proofs for Section A.7.1

Proof of Proposition A.6.

1. If $\emptyset \not\in\uparrow \Gamma$, then $\emptyset \not\in \Gamma$ because $\Gamma \subseteq\uparrow \Gamma$. Conversely, if $\emptyset \in\uparrow \Gamma$ then there is $E \in \Gamma$ with $E \subseteq \emptyset$, i.e., $\emptyset \in \Gamma$. Colloquially, if a contradiction is logically entailed from a given information collection then it has to contain a contradiction.

2. Suppose that Γ satisfies Consistency. Suppose to the contrary that there are $E, F \in \Gamma$ with $E \cap F = \emptyset$. Then, since $F \subseteq E^c$, it follows that $E, E^c \in\uparrow \Gamma$, a contradiction. Conversely, if $E \in\uparrow$ then there is $E' \in \Gamma$ such that $E' \subseteq E$. If $E^c \in\uparrow \Gamma$ then there is $F' \in \Gamma$ such that $F' \subseteq E^c$ and thus $E' \cap F' \subseteq E \cap E^c$, which is impossible. Hence, $E^c \not\in\uparrow \Gamma$.

3. If $\Gamma \neq \emptyset$ then there is $E \in \Gamma$. Since $E \subseteq \Omega$, we have $\Omega \in\uparrow \Gamma$. Conversely, if $\Omega \in\uparrow \Gamma$ then there is $E \in \mathcal{D}$ with $E \in \Gamma$, and thus $\Gamma \neq \emptyset$.

4. Suppose that Γ satisfies Non-empty λ-Conjunction. Let $\mathcal{F} \subseteq \Gamma$ be with $0 < |\mathcal{F}| < \lambda$. Then, since $\Gamma \subseteq\uparrow \Gamma$, we have $\bigcap \mathcal{F} \in\uparrow \Gamma$, i.e., there is $F \in \Gamma$ with $F \subseteq \bigcap \mathcal{F}$. Conversely, let $\mathcal{F} \subseteq\uparrow \Gamma$ be with $0 < |\mathcal{F}| < \lambda$. For each $F \in \mathcal{F}$, there is $F' \in \Gamma$ such that $F' \subseteq F$. Let $\mathcal{F}'$ be a collection of such F' for each $F \in \mathcal{F}$. Then, there is $F \in \mathcal{F}$ with $F \subseteq \bigcap \mathcal{F}' \subseteq \bigcap \mathcal{F}$, i.e., $\bigcap \mathcal{F} \in\uparrow \Gamma$.

□

Proof of Proposition A.7.

1. If $\mathcal{B}_i$ is reflexive then it is trivial that $\omega \in E$ for any $\omega \in \Omega$ and $E \in \mathcal{B}_i(\omega)$. The converse is also clear because, if $\omega \in \Omega$ and $E \in\uparrow \mathcal{B}_i(\omega)$ then there is $F \in \mathcal{B}_i(\omega)$ with $\omega \in F \subseteq E$.

2. Suppose that $\mathcal{B}_i$ is transitive. Take any $(\omega, E) \in \Omega \times \mathcal{D}$ with $E \in \mathcal{B}_i(\omega)$. It then follows from the supposition that $\{\omega' \in \Omega \mid E \in\uparrow \mathcal{B}_i(\omega')\} \in\uparrow \mathcal{B}_i(\omega)$. Thus there is $F \in \mathcal{B}_i(\omega)$ such that $F \subseteq \{\omega' \in \Omega \mid E \in\uparrow \mathcal{B}_i(\omega')\}$, clearly showing that if $\omega' \in F$ then there is $E' \in \mathcal{B}_i(\omega')$ such that $E' \subseteq E$. Conversely, suppose

that $E \in\uparrow \mathcal{B}_i(\omega)$, i.e., suppose that there is $F \in \mathcal{B}_i$ such that $F \subseteq E$. For this $F \in \mathcal{B}_i(\omega)$, there is $F' \in \mathcal{B}_i(\omega)$ such that if $\omega' \in F'$ then $F \in\uparrow \mathcal{B}_i(\omega')$. Thus, we obtain

$$F' \subseteq \{\omega' \in \Omega \mid F \in\uparrow \mathcal{B}_i(\omega')\} \subseteq \{\omega' \in \Omega \mid E \in\uparrow \mathcal{B}_i(\omega')\}.$$

Since $F' \in \mathcal{B}_i(\omega)$, it follows that $\{\omega' \in \Omega \mid E \in\uparrow \mathcal{B}_i(\omega')\} \in\uparrow \mathcal{B}_i(\omega)$.

3. First observe that, for any $E \in \mathcal{D}$, we have $E \not\in \mathcal{B}_i(\omega)$ iff $E^c \cap F \neq \emptyset$ for all $F \in \mathcal{B}_i(\omega)$. Thus, $\mathcal{B}_i$ is Euclidean iff the following holds: if there is $(\omega, E) \in \Omega \times \mathcal{D}$ such that $E^c \cap F \neq \emptyset$ for all $F \in \mathcal{B}_i(\omega)$, then there is $F' \in \mathcal{B}_i(\omega)$ such that $F' \subseteq \{\omega' \in \Omega \mid E \not\in\uparrow \mathcal{B}_i(\omega')\}$. This last part is equivalent to the following: if $\omega' \in F'$ then $E^c \cap F \neq \emptyset$ for any $F \in \mathcal{B}_i(\omega')$.

□

Proof of Proposition A.8. 1. (a) If $\omega \in K_{\mathcal{B}_i}(\emptyset)$, then $\emptyset \in\uparrow \mathcal{B}_i(\omega)$. Hence, $K_{\mathcal{B}_i}(\emptyset)$ inherits No-Contradiction Axiom.

(b) We show that $K_{\mathcal{B}_i}$ satisfies Consistency. If $\omega \in K_{\mathcal{B}_i}(E)$ then $E \in\uparrow \mathcal{B}_i(\omega)$. Since $E^c \not\in\uparrow \mathcal{B}_i(\omega)$, we have $\omega \in (\neg K_{\mathcal{B}_i})(E^c)$.

(c) Consider Non-empty λ-Conjunction. Take any $\mathcal{F} \subseteq \mathcal{D}$ with $0 < |\mathcal{F}| < \kappa$. If $\omega \in \bigcap_{F \in \mathcal{F}} K_{\mathcal{B}_i}(F)$ then $F \in\uparrow \mathcal{B}_i(\omega)$ for all $F \in \mathcal{F}$. Thus, $\bigcap \mathcal{F} \in\uparrow \mathcal{B}_i(\omega)$, i.e., $\omega \in K_{\mathcal{B}_i}(\bigcap \mathcal{F})$.

(d) Consider Necessitation. For any $\omega \in \Omega$, there is $\Omega \in\uparrow \mathcal{B}_i(\omega)$ so that $\omega \in K_i(\Omega)$.

(e) If $\mathcal{B}_i$ is reflexive, then for any $\omega \in K_{\mathcal{B}_i}(E)$, we have $E \in\uparrow \mathcal{B}_i(\omega)$ and thus $\omega \in E$. Thus, Truth Axiom is established.

(f) Consider Positive Introspection. If $\omega \in K_{\mathcal{B}_i}(E)$, then $E \in\uparrow \mathcal{B}_i(\omega)$. Now, we have $\{\omega' \in \Omega \mid E \in\uparrow \mathcal{B}_i(\omega')\} \in\uparrow \mathcal{B}_i(\omega)$, i.e., $\omega \in K_{\mathcal{B}_i} K_{\mathcal{B}_i}(E)$.

(g) Consider Negative Introspection. If $\omega \in (\neg K_{\mathcal{B}_i})(E)$, then $E \not\in\uparrow \mathcal{B}_i(\omega)$. Now, we have $\{\omega' \in \Omega \mid E \not\in\uparrow \mathcal{B}_i(\omega')\} \in\uparrow \mathcal{B}_i(\omega)$, i.e., $\omega \in (\neg K_{\mathcal{B}_i}) K_{\mathcal{B}_i}(E)$.

(h) The Kripke property of $K_{\mathcal{B}_i}$ amounts to $E \in\uparrow \mathcal{B}_i(\omega)$ (i.e., $\omega \in K_{\mathcal{B}_i}(E)$) whenever $(\bigcap\{F \in \mathcal{D} \mid \omega \in K_{\mathcal{B}_i}(F)\} =) \bigcap \uparrow \mathcal{B}_i(\omega) \subseteq E$. This simply follows from the Kripke property of $\mathcal{B}_i$.

Finally, we have $\mathcal{B}_{K_{\mathcal{B}_i}}(\omega) = \{E \in \mathcal{D} \mid \omega \in K_{\mathcal{B}_i}(E)\} = \{E \in \mathcal{D} \mid E \in\uparrow \mathcal{B}_i(\omega)\} =\uparrow \mathcal{B}_i(\omega)$ for all $\omega \in \Omega$.

2. Let K_i be a given knowledge operator and let $\mathcal{B}_i$ be a generator of K_i.

(a) Consider No-Contradiction Axiom. Fix $\omega \in \Omega$. Since $\omega \not\in K_i(\emptyset)$, we have $\emptyset \not\in\uparrow \mathcal{B}_i(\omega)$.

(b) Consider Consistency. Take any $\omega \in \Omega$ and $E \in\uparrow \mathcal{B}_i(\omega)$. Since $\omega \in K_i(F) \subseteq (\neg K_i)(E^c)$, we have $E^c \not\in\uparrow \mathcal{B}_i(\omega)$.

(c) Consider Non-empty λ-Conjunction. Let $\mathcal{F}$ be a subset of $\mathcal{D}$ with $0 < |\mathcal{F}| < \lambda$. If $F \in\uparrow \mathcal{B}_i(\omega)$ for all $F \in \mathcal{F}$ then $\omega \in \bigcap_{F\in\mathcal{F}} K_i(F) \subseteq K_i(\bigcap \mathcal{F})$. Thus, $\bigcap \mathcal{F} \in\uparrow \mathcal{B}_i(\omega)$.

(d) Consider Necessitation. For any $\omega \in \Omega$, we have $\omega \in K_i(\Omega)$, i.e., $\Omega \in\uparrow \mathcal{B}_i(\omega)$.

(e) Suppose that K_i satisfies Truth Axiom. Fix $\omega \in \Omega$ and $E \in\uparrow \mathcal{B}_i(\omega)$. Then, we have $\omega \in K_i(E) \subseteq E$.

(f) Consider Positive Introspection. Fix $\omega \in \Omega$ and $E \in\uparrow \mathcal{B}_i(\omega)$. Then, $\omega \in K_i(E) \subseteq K_i K_i(E)$, and thus $\{\omega' \in \Omega \mid E \in\uparrow \mathcal{B}_i(\omega')\} \in\uparrow \mathcal{B}_i(\omega)$.

(g) Suppose that K_i satisfies Negative Introspection. Fix $\omega \in \Omega$ and $E \notin\uparrow \mathcal{B}_i(\omega)$. Then, we have $\omega \in (\neg K_i)(E) \subseteq K_i(\neg K_i)(E)$, and thus $\{\omega' \in \Omega \mid E \notin\uparrow \mathcal{B}_i(\omega')\} \in\uparrow \mathcal{B}_i(\omega)$.

(h) Consider the Kripke property. Observe that $\bigcap \uparrow \mathcal{B}_i(\omega) = \bigcap\{F \in \mathcal{D} \mid \omega \in K_i(F)\}$. The Kripke property of K_i implies that $\bigcap \uparrow \mathcal{B}_i(\omega) \subseteq E$ iff $E \in\uparrow \mathcal{B}_i(\omega)$, establishing the Kripke property of $\mathcal{B}_i$.

Finally, for each $E \in \mathcal{D}$, we have

$$K_{\mathcal{B}_{K_i}}(E) = \{\omega \in \Omega \mid E \in\uparrow \mathcal{B}_{K_i}(\omega)\} = \{\omega \in \Omega \mid \omega \in K_i(E)\} = K_i(E).$$

□

Proof of Proposition A.9. Take $E' \in \mathcal{B}'_i(\varphi(\omega))$. Since $E' \subseteq E'$, we have $\varphi(\omega) \in K'_i(E')$. Then, $\omega \in \varphi^{-1}(K'_i(E')) = K_i(\varphi^{-1}(E'))$. Now, there is $E \in \mathcal{B}_i(\omega)$ such that $E \subseteq \varphi^{-1}(E')$ and hence $\varphi(E) \subseteq \varphi(\varphi^{-1}(E')) \subseteq E$.

Next, take $E \in \mathcal{B}_i(\omega)$ and $E' \in \mathcal{D}'$ with $\varphi(E) \subseteq E'$. Then, since $E \subseteq \varphi^{-1}(\varphi(E)) \subseteq \varphi^{-1}(E')$, we have $\omega \in K_i(\varphi^{-1}(E')) = \varphi^{-1}(K'_i(E'))$. Thus, we get $\varphi(\omega) \in K'_i(E')$. Then, there is $F' \in \mathcal{B}_i(\varphi(\omega))$ such that $F' \subseteq E'$.

Suppose that (2) holds. If $\omega \in K_i(\varphi^{-1}(E'))$ then there is $E \in \mathcal{B}_i(\omega)$ such that $E \subseteq \varphi^{-1}(E')$, i.e., $\varphi(E) \subseteq \varphi(\varphi^{-1}(E')) \subseteq E'$. By (2), there is $F' \in \mathcal{B}'_i(\varphi(\omega))$ such that $F' \subseteq E'$. Hence, $\varphi(\omega) \in K'_i(E')$, i.e., $\omega \in \varphi^{-1}(K'_i(E'))$.

Suppose that (1) holds. If $\omega \in \varphi^{-1}(K'_i(E'))$ then $\varphi(\omega) \in K'_i(E')$. There is $F' \in \mathcal{B}'_i(\varphi(\omega))$ such that $F' \subseteq E'$. It follows from (1) that there is $F \in \mathcal{B}_i(\omega)$ such that $\varphi(F) \subseteq F'$. Thus, $\varphi(F) \subseteq E'$ and hence $F \subseteq \varphi^{-1}(\varphi(F)) \subseteq \varphi^{-1}(E')$. Then, we get $\omega \in K_i(\varphi^{-1}(E'))$. □

A.7.3 Agreement Theorem

Here, we extend Samet's [77] qualitative agreement theorem to our generalized framework.[A.7] The non-probabilistic agreement theorem says that players cannot have common knowledge of their actions unless their actions are indeed identical. We extend

[A.7] See also the references in Samet [77] for other qualitative generalizations of the agreement theorem.

Samet's [77] qualitative agreement theorem independently of assumptions on players' knowledge.

Throughout, fix a κ-complete algebra $(\Omega, \mathcal{D})$, where κ is an infinite (regular) cardinal or $\kappa = \infty$. Let $(K_i)_{i \in I}$ be players' knowledge operators on $\mathcal{D}$. Let $\mathcal{A}$ be a non-empty set of *actions* endowed with a κ-complete algebra $\mathcal{D}_{\mathcal{A}}$. We assume that $\{a\} \in \mathcal{D}_{\mathcal{A}}$ for all $a \in \mathcal{A}$. An action function of player i is a κ-measurable mapping $\alpha_i : (\Omega, \mathcal{D}) \to (\mathcal{A}, \mathcal{D}_{\mathcal{A}})$. We denote by $[\alpha_i = a] := \{\omega \in \Omega \mid \alpha_i(\omega) = a\}$ the event that player i takes an action a.

We denote by $[j \succcurlyeq i]$ the set of states at which player j is *at least as knowledgeable as* player i: $[j \succcurlyeq i] := \bigcap_{E \in \mathcal{D}} ((\neg K_i)(E) \cup K_j(E))$. The set of states at which players i and j are *equally knowledgeable* is defined by $[j \sim i] := [j \succcurlyeq i] \cap [i \succcurlyeq j]$. A player $i \in I$ is said to be an *epistemic dummy* if $[j \succcurlyeq i] = \Omega$ for all $j \in I$.

We make the following two assumptions, by slightly modifying Samet's [77] corresponding assumptions. The first is the *Interpersonal Sure-Thing Principle* (ISTP). Suppose that there is an event E indicating (i) that a player j is at least as knowledgeable as a player i and (ii) that j's action is a. If player i knows E irrespective of a particular form of E in the sense formalized below, we assume that she also takes the action a. Formally, fix any $i, j \in I$ and $a \in \mathcal{A}$. Suppose that there is $E \in \mathcal{D}$ such that $E \subseteq [j \succcurlyeq i] \cap [\alpha_j = a]$ and that $\omega \in K_i(F)$ for all $F \in \mathcal{D}$ with $E \subseteq F \subseteq [j \succcurlyeq i] \cap [\alpha_j = a]$. Then $\omega \in [\alpha_i = a]$.

As a remark, if two agents i and j, who know their own action function, are equally knowledgeable at any state, then the ISTP implies that they take the same action at any state. Indeed, take any $\omega \in \Omega$ and let $a = \alpha_j(\omega)$. The ISTP, together with the assumption that each player knows her own action function, imply that $\omega \in K_j([\alpha_j = a]) = K_i([\alpha_j = a]) = K_i([\alpha_j = a] \cap [j \succcurlyeq i]) \subseteq [\alpha_i = a]$.

The second assumption is the *ISTP Expandability*. Let $\langle (\Omega, \mathcal{D}), (K_i)_{i \in I}, (\alpha_i)_{i \in I} \rangle$ be given, and let i^* be the epistemic dummy whose knowledge corresponds to the common knowledge among I.[A.8] The ISTP Expandability says that there exists an action function α_{i^*} of the epistemic dummy i^* such that the extended action function profile $(\alpha_i)_{i \in I \cup \{i^*\}}$ satisfies the ISTP. Now, we formalize the qualitative agreement theorem.

Proposition A.10 (Qualitative Agreement Theorem). *Let $\langle (\Omega, \mathcal{D}), (K_i)_{i \in I}, (\alpha_i)_{i \in I} \rangle$ be an ISTP expandable. If, for every $i \in I$, it is commonly known among I at a state ω that $\alpha_i(\omega) = a_i$, then players' actions $(a_i)_{i \in I}$ are identical. That is, if $\omega \in \bigcap_{i \in I} C([\alpha_i = a_i])$ then $a_i = a_j$ for all $i, j \in I$.*

Proof of Proposition A.10. Define an epistemic dummy i^* whose knowledge coincides with the common knowledge. Now, by the ISTP Expandability, there is an action function α_{i^*} such that $(\alpha_i)_{i \in I \cup \{i^*\}}$ satisfies the ISTP. Suppose that $\omega \in \bigcap_{i \in I} C([\alpha_i = a_i]) = \bigcap_{i \in I} K_{i^*}([\alpha_i = a_i])$. It follows from the ISTP that $\omega \in K_{i^*}([\alpha_i = a_i]) =$

[A.8]The knowledge of the epistemic dummy may fail to satisfy some assumptions (e.g., Negative Introspection) imposed on individual players.

$K_{i^*}([i \succcurlyeq i^*] \cap [\alpha_i = a_i]) \subseteq [\alpha_{i^*} = a_i]$ for each $i \in I$. Thus, we obtain $\alpha_{i^*}(\omega) = a_i = a_j$ for all $i, j \in I$. $\square$

It is wothwhile noting that Proposition A.10 holds irrespective of the cardinality of Ω. Samet [76] shows that Aumann's [3] probabilistic agreement theorem may fail to hold in some reflexive and transitive possibility correspondence model when Ω is uncountable. Thus, Aumann's [3] agreement theorem is not necessarily a special case of the qualitative agreement theorem in non-partitional spaces.

We also mention another difference between partitional and non-partitional cases. Henceforth, suppose that every agent knows her own action function. In partitional cases, equally knowledgeable agents at a particular state take the same action at that state under the ISTP (Samet [77, Proposition 1]). However, it is not necessarily the case in non-partitional models.[A.9]

Let $I = \{1, 2\}$, $\Omega = \{\omega_1, \omega_2, \omega_3\}$, and $\mathcal{D} = \mathcal{P}(\Omega)$. We define player 1's knowledge by her possibility correspondence: $b_1(\omega_1) = \{\omega_1\}$ and $b_1(\omega_j) = \Omega$ for $j \in \{2, 3\}$. Likewise, we define player 2's knowledge by $b_2(\omega_j) = \Omega$ and $b_2(\omega_3) = \{\omega_3\}$ for $j \in \{1, 2\}$.

For actions, we let $\mathcal{A} = \{a_1, a_2\}$, $\mathcal{D}_{\mathcal{A}}$, and $\alpha_i(\cdot) = a_i$ for each $i \in I$. Then, each player i knows her action function $\alpha_i : (\Omega, \mathcal{D}) \to (\mathcal{A}, \mathcal{D}_{\mathcal{A}})$. Also, it can be seen that the ISTP is trivially satisfied (i.e., $K_i([j \succcurlyeq i] \cap [\alpha_j = a_j]) = \emptyset$).

It can be seen that, while players 1 and 2 are equally knowledgeable at ω_2 ($[1 \sim 2] = \{\omega_2\}$), there is no state at which they take the same action (i.e., $\bigcup_{a \in \{a_1, a_2\}}([\alpha_1 = a] \cap [\alpha_1 = a]) = \emptyset$). In this particular example, both players are equally knowledgeable at ω_2 because they "mistakenly" ignore their knowledge. If they were to obey partitional knowledge, then player 1's partition cell at ω_2 would be $\{\omega_2, \omega_3\}$ and that of player 2 would be $\{\omega_1, \omega_2\}$.

A.7.4 Relation to a Model of Knowledge in Psychology

Here, we state connections between a model of knowledge in psychology studied by Doignon and Falmagne [22, 23] and our framework.[A.10] Doignon and Falmagne [22, 23] study the knowledge of an agent regarding subsets of Ω, which, in their work, consists of questions (note that we use our corresponding notations in order to make it easier to see the connections). An agent's knowledge is modeled as a pair $(\Omega, \mathcal{J})$, where $\emptyset, \Omega \in \mathcal{J} \subseteq \mathcal{P}(\Omega)$ and $\mathcal{J}$ is closed under arbitrary union. Thus, in terms of our framework, knowledge is defined on $\mathcal{D} = \mathcal{P}(\Omega)$ and $\mathcal{J}$ satisfies the maximality property as well as Necessitation.[A.11] In their work, each set $E \in \mathcal{J}$ is interpreted as a set of questions

[A.9]Note that this statement is different from the previously stated result: if two agents are equally knowledgeable at *all* states then they take the same action at all states.

[A.10]It is worth pointing out that Doignon and Falmagne [23] document empirical procedures for asessing the knowledge of an agent in the psychology literature.

[A.11]Recall that the maximality property is equivalent to the closure under arbitrary union on a complete algebra (Proposition 5).

that an agent is capable of solving.

We make the following two connections. First, Doignon and Falmagne [22, 23] show one way to represent an agent's knowledge is to construct $\mathcal{J}_{\preccurlyeq}$ from a pre-ordered set $(\Omega, \preccurlyeq)$. They call the pre-order a surmise relation, and it is interpreted as follows: for any questions $\omega, \omega' \in \Omega$, it is said that ω' is surmised from ω (denoted $\omega \preccurlyeq \omega'$) iff it can be surmised that, from observing a correct response to question ω, a correct response would be given to question ω'. Thus, a question ω' is "at least as informative as" ω for assessing an agent's knowledge when $\omega \preccurlyeq \omega'$. The collection $\mathcal{J}_{\preccurlyeq}$ represents an agent's knowledge in the sense that each $E \in \mathcal{J}_{\preccurlyeq}$ is an upper set with respect to $\preccurlyeq$ (i.e., $\omega' \in E$ for all $\omega \in E$ and $\omega \preccurlyeq \omega'$). It can be seen that $\mathcal{J}_{\preccurlyeq}$ is closed under arbitrary union and intersection (with $\emptyset, \Omega \in \mathcal{J}_{\preccurlyeq}$). Thus, in our framework, knowledge derived from $\mathcal{J}_{\preccurlyeq}$ satisfies: Truth Axiom, Monotonicity, Positive Introspection, Necessitation, and (Non-empty) Conjunction.

Observe that the above properties characterize the reflexive and transitive possibility correspondence models in ∞-knowledge spaces.[A.12] Suppose that a pre-ordered set $(\Omega, \preccurlyeq)$ is given. Viewing Ω as a set of states of the world, each upper contour set of ω with respect to $\preccurlyeq$ defines the reflexive and transitive possibility correspondence $b_{\preccurlyeq}(\omega) = \{\omega' \in \Omega \mid \omega \preccurlyeq \omega'\}$. In relation to modal logic, the surmise relation $\preccurlyeq$ is a (reflexive and transitive) accessibility relation. An upper set $E \in \mathcal{J}_{\preccurlyeq}$ with respect to $\preccurlyeq$ is a self-evident event E in terms of $b_{\preccurlyeq}$: $b_{\preccurlyeq}(\omega) \subseteq E$ (i.e., $\omega \in K_{b_{\preccurlyeq}}(E)$) for all $\omega \in E$.

Second, Doignon and Falmagne [22, 23] introduce a surmise system $(\Omega, \mathcal{B})$ as another way to induce an agent's knowledge. A surmise mapping is a mapping $\mathcal{B} : \Omega \to \mathcal{P}(\mathcal{P}(\Omega))$ with the following interpretation. Each $\mathcal{B}(\omega)$ is interpreted as encoding all possible (not necessarily unique) ways of inferring a correct response to the question ω. Put differently, if an agent is capable of solving a question ω, then there exists $E \in \mathcal{B}(\omega)$ such that she is capable of solving all the questions in E. With this interpretation, Doignon and Falmagne [22, 23] define $\mathcal{J}_{\mathcal{B}}$ as a collection of $E \in \mathcal{P}(\Omega)$ such that for all $\omega \in E$, there is $F \in \mathcal{B}(\omega)$ such that $F \subseteq E$. Thus, a surmise system is closely related to an information correspondence.[A.13]

[A.12]Note that the possibility relation between states is equivalent to the informativeness relation in this situation. See Proposition 12.

[A.13]Doignon and Falmagne [22, 23], however, impose the following minimality condition: if $E, F \in \mathcal{B}(\omega)$ satisfy $E \subseteq F$ then $E = F$. Doignon and Falmagne [22, Theorem 3.7] establish the equivalence between a surmise system $\mathcal{B}$ and a collection $\mathcal{J}_{\mathcal{B}}$ of subsets when an underlying set Ω is finite. More generally, Doignon and Falmagne [23, Theorem 3.10] establish the equivalence between a surmise system and a collection which are "granular" for an arbitrary underlying set. In our framework, we demonstrate the equivalence between a reflexive and transitive information correspondence and a self-evident collection when a player's knowledge satisfies Truth Axiom, Positive Introspection, and Monotonicity without imposing the minimality condition.